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THE FLAT STRIP THEOREM FAILS FOR SURFACES WITH NO CONJUGATE POINTS

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ABSTRACT. A compact C^∞ surface with no conjugate points is constructed so that there are two homotopic closed geodesics that do not bound a flat annulus.

One of the basic rigidity properties of manifolds with nonpositive curvature is the

Flat Strip Theorem [EON]. *Let β and γ be geodesics in a simply connected manifold with nonpositive curvature. Suppose that β and γ have finite Hausdorff distance. Then β and γ are the edges of a flat strip, i.e., an isometrically and totally geodesically embedded copy of $I \times \mathbf{R}$.*

Two natural generalizations of nonpositive curvature are the no focal point and no conjugate point properties. These can be described in terms of Jacobi fields [OS1, Proposition 4]. A complete Riemannian manifold has

- (i) nonpositive curvature if and only if $\langle Y, Y\rangle'' \geq 0$ for every Jacobi field Y;
- (ii) no focal points if and only if $\langle Y, Y\rangle'(t) > 0$ whenever $t > 0$ and Y is a nontrivial Jacobi field with $Y(0) = 0$;
- (iii) no conjugate points if and only if $\langle Y, Y\rangle(t) > 0$ whenever $t > 0$ and Y is a nontrivial Jacobi field with $Y(0) = 0$.

Gulliver [G] has given examples that show (iii) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i).

The flat strip theorem holds for manifolds with no focal points [OS2, E]. The present paper shows that it fails for surfaces with no conjugate points. We construct a compact C^∞ surface S with no conjugate points that contains a closed annulus that is foliated by homotopic closed geodesics, but is not flat. A lift of this annulus to the universal cover is a nonflat strip bounded by two geodesics with finite Hausdorff distance.

Here is a brief description of S. Consider a torus of revolution T that is flat except for one small bulge. The torus T is foliated by closed meridian

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geodesics. Let Σ_1 be a thin annulus that is a union of meridians. A C^1 approximation to S can be constructed by taking a surface with constant curvature -1 that contains a simple closed geodesic with the same length as the meridians of T, cutting the surface along this geodesic and then gluing in Σ_1. One could think of S as being obtained from this surface by carefully smoothing the metric near $\partial\Sigma_1$. Instead we construct a C^∞ Riemannian metric g on $S^1 \times \mathbf{R}$ so that

(i) the curve $\sigma = S^1 \times \{0\}$ is a geodesic;
(ii) the natural coordinates $s \in S^1$ and $y \in \mathbf{R}$ are Fermi coordinates for σ;
(iii) σ has a neighbourhood Σ isometric to Σ_1;
(iv) the curvature is -1 outside a neighbourhood of Σ.

The two ends of $(S^1 \times \mathbf{R}, g)$ are then replaced by compact ends with curvature -1, using a standard construction. Property (ii) of g makes it easy to prescribe the curvature near $\partial\Sigma$. Note finally that S has focal points because there are periodic Jacobi fields along σ.

1. Preliminaries

Suppose we define a metric on a surface with respect to the coordinates s and y by

$$\begin{pmatrix} g_{ss} & g_{sy} \\ g_{ys} & g_{yy} \end{pmatrix} = \begin{pmatrix} \psi^2(s, y) & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\psi > 0$ and ψ is the solution of the initial value problem

$$\psi_{yy}(s, y) + K(s, y)\psi(s, y) = 0, \qquad \psi(s, 0) = 1, \qquad \psi_y(s, 0) = 0. \tag{1.2}$$

Then the curve σ defined by $y = 0$ is a geodesic for which s and y are Fermi coordinates; s measures arclength along σ and $|y|$ is distance from σ. The curvature at the point with coordinates (s, y) is $K(s, y)$. In our case, σ will be a closed geodesic of length 2π, so we take $s \in S^1$. If $s_1, s_2 \in S^1$, (s_1, s_2) and $[s_1, s_2]$ will denote the open and closed arcs respectively from s_1 to s_2 in the anticlockwise direction.

2. Construction of T and S

Let $\alpha = 1/110$. The torus T will be a warped product $S^1 \times_f S^1$ (warped products are described in [BON]), where f is a C^∞ function on S^1 such that $f(s) = 1$ if $s \in [e^{i\alpha}, e^{-i\alpha}]$ and $1 < f(s) < 9$ if $s \in (e^{-i\alpha}, e^{i\alpha})$. Thus T is a torus of revolution with the curves $S^1 \times_f \{s\}$, $s \in S^1$, as meridian geodesics. Let σ_1 be the geodesic $S^1 \times_f \{1\}$. We choose f so that the curvature K_T of T satisfies $|K_T| \leq 10^{-6}$ throughout T. It is clear from this that there is a diffeomorphism $\pi : S^1 \times [-1/10, 1/10] \to \{p \in T : \mathrm{dist}(p, \sigma_1) \leq 1/10\}$ such that (s, y) are the Fermi coordinates with respect to σ_1 for the point $\pi(s, y)$.

We now choose the curvature $K(s, y)$ that we want for the metric g on $S^1 \times \mathbf{R}$. Choose a C^∞ function $h : S^1 \to [\alpha, 9\alpha]$ such that

$$h(s) = \begin{cases} 9\alpha & \text{if } s \in [e^{-i\alpha}, e^{i\alpha}], \\ \alpha & \text{if } s \in [e^{2i\alpha}, e^{-2i\alpha}]. \end{cases}$$

Let $H = \{(s, y) \in S^1 \times \mathbf{R} : |y| \leq h(s)\}$ and $H_\alpha = \{(s, y) \in S^1 \times \mathbf{R} : |y| \leq h(s) + \alpha\}$. Choose a C^∞ function $\theta : S^1 \times \mathbf{R} \to [0, 1]$ with $\theta = 1$ on H and $\theta = 0$ outside H_α. Define $K(s, y)$ by

$$K(s, y) = \begin{cases} K_T(\pi(s, y)) & \text{if } (s, y) \in H, \\ \theta(s, y)K_T(\pi(s, y)) - \{1 - \theta(s, y)\} & \text{if } (s, y) \in H_\alpha \setminus H, \\ -1 & \text{if } (s, y) \notin H_\alpha. \end{cases}$$

See Figure 1. Observe that K_T vanishes in the region $[e^{i\alpha}, e^{-i\alpha}] \times_f S^1$ and

(2.1) $\qquad -1 \leq K \leq 10^{-6}$ everywhere;

(2.2) $\qquad K \leq 0$ outside $(e^{-i\alpha}, e^{i\alpha}) \times (-10\alpha, 10\alpha)$;

(2.3) $\qquad K = -1$ outside $((e^{-2i\alpha}, e^{2i\alpha}) \times (-10\alpha, 10\alpha)) \cup (S^1 \times (-2\alpha, 2\alpha))$.

Finally g is the metric represented with respect to the s-y coordinates by the matrix (1.1) in which ψ satisfies (1.2). It is easily shown using (2.1) and

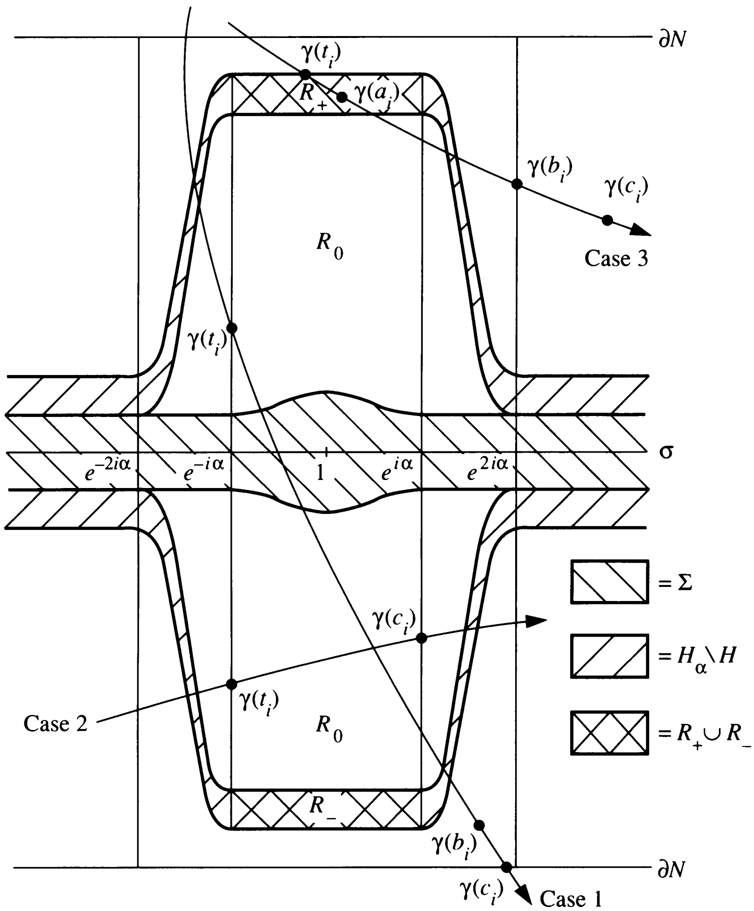

FIGURE 1

(2.3) that $\psi(s, y) > 0$ for all (s, y). One can even see that $\psi(s, y) \to \infty$ uniformly as $|y| \to \infty$. The construction ensures that the curve $\sigma(t) = (e^{it}, 0)$ is a unit speed geodesic for the metric g and (s, y) are Fermi coordinates for σ throughout $S^1 \times \mathbf{R}$.

Let $\Sigma_1 = S^1 \times_f [e^{-i\alpha}, e^{i\alpha}]$, so Σ_1 is a neighbourhood of σ_1 foliated by meridian geodesics. It is clear from the choice of f and g that $\Sigma_1 \subseteq \pi(H)$. Since $K = K_T \circ \pi$ on H, we see that $\Sigma \stackrel{\text{def}}{=} \pi^{-1}(\Sigma_1)$ is isometric to Σ_1.

We now outline the procedure for replacing the ends of $(S^1 \times \mathbf{R}, g)$ with compact ends. It follows a similar construction in [BBB]. If we cut a component of $(S^1 \times \mathbf{R}) \setminus N$ along the geodesic $\{1\} \times \mathbf{R}$ we obtain a fan-shaped subset of the Poincaré disc shown in Figure 2.

This subset is bounded by two geodesic rays η and η' and a curve corresponding to a component of ∂N. For any large enough n, it is possible to draw a sequence of $4n + 1$ hyperbolic geodesic segments $d'_n, c'_n, d'_{n-1}, \dots, d'_1, c'_1, d_0, c''_1, d''_1, \dots, c''_n, d''_n$ such that

- d'_n and d''_n are orthogonal to η' and η'' respectively;
- adjacent geodesics in the sequence are orthogonal;
- c'_i and c''_i have the same length for $1 \le i \le n$.

By identifying η' with η'' and c'_i with c''_i, $1 \le i \le n$, we obtain a hyperbolic metric on the sphere with $n + 2$ punctures. One of the holes is bounded by a copy of a component of ∂N and the others by closed geodesics $d_0, \dots, d_n$. Now we can adjoin handles with curvature -1 along $d_0, \dots, d_n$.

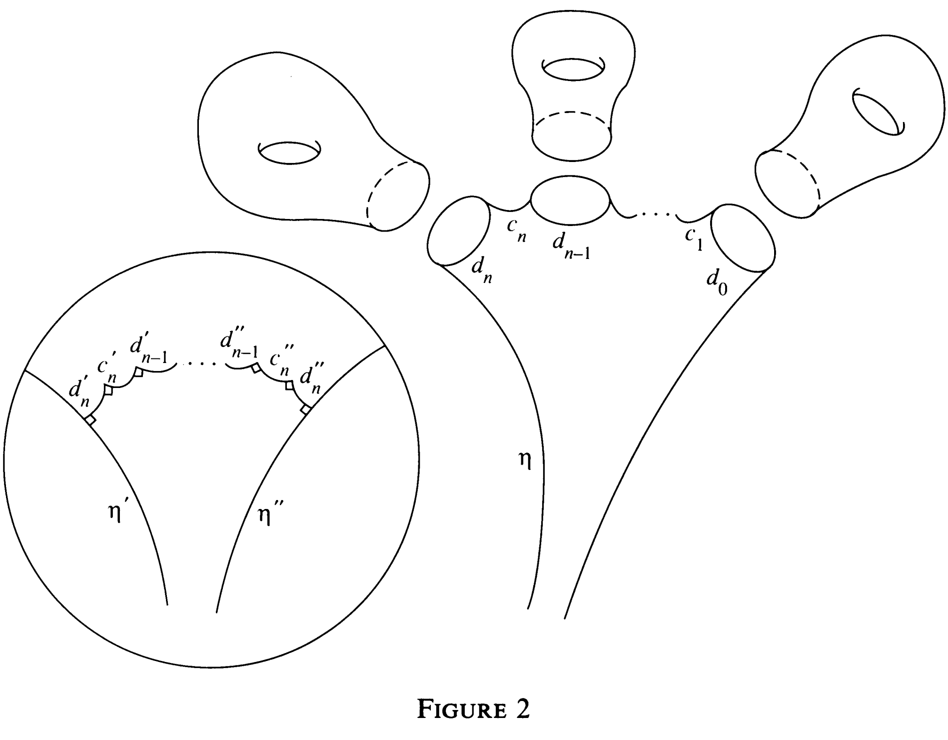

FIGURE 2

3. Geodesics in N

Let $N = S^1 \times [-1/10, 1/10]$. Note that $H_\alpha \subseteq N$, since $10\alpha < 1/10$.

3.1. **Lemma.** *For any* $(s, y) \in N$ *and any unit vector* $u \in T_{(s,y)}N$,

(i) $.99 \le \|\frac{\partial}{\partial s}(s, y)\| \le 1.01$;

(ii) $\|\nabla_u \frac{\partial}{\partial y}\| \le 0.11$.

Proof. Note that $|K(s, y)| \le 1$, by (2.1). Since $|y| \le 1/10$, it follows from (1.2) and the Sturm Comparison Theorem that

$$.99 \le \cos(1/10) \le \psi(s, y) \le \cosh(1/10) \le 1.01.$$

This proves (i), since $\psi = \|\frac{\partial}{\partial s}\|$.

Since the level curves for s are the geodesics orthogonal to σ and $\frac{\partial}{\partial y}(s, 0)$ is a smooth field of unit vectors normal to σ,

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}(s, y) = 0 \quad \text{and} \quad \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial y}(s, 0) = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial s}(s, 0) = 0.$$

Since $\frac{\partial}{\partial s}$ is a Jacobi field along each geodesic $s =$ const,

$$\nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial s} = -R(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial y})\tfrac{\partial}{\partial y},$$

where R is the curvature tensor for g. Thus if $u = a\frac{\partial}{\partial s}(s, y) + b\frac{\partial}{\partial y}(s, y)$, we have

$$\|\nabla_u \tfrac{\partial}{\partial y}\| = \|a\nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial y}\| \le |a| \int_0^y \|R(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial y})\tfrac{\partial}{\partial y}(s, z)\|\, dz.$$

Since $\langle R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y})\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle = 0$ and $\|\frac{\partial}{\partial y}\| = 1$,

$$\begin{aligned}\|R(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial y})\tfrac{\partial}{\partial y}\| &= \|\tfrac{\partial}{\partial s}\|^{-1}|\langle R(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial y})\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial s}\rangle| \\ &= \|\tfrac{\partial}{\partial s}(s, y)\| \cdot |K(s, y)| \le \|\tfrac{\partial}{\partial s}(s, y)\|.\end{aligned}$$

Thus $\|\nabla_u \frac{\partial}{\partial y}\| \le |a| \cdot |y| \cdot \sup_N \|\frac{\partial}{\partial s}(s, y)\| \le (0.99)^{-1} \cdot (1/10) \cdot 1.01 \le 0.11$, by (i). □

3.2. **Corollary.** *Let* γ *be a maximal unit speed geodesic in* N *that makes angle* $\pi/4$ *with the* s *and* y *directions at* $\gamma(0)$. *Then the angle between* γ *and the* s *direction is always between* $\pi/6$ *and* $\pi/3$.

Proof. Since $\frac{\partial}{\partial y}$ is the unit vector field in the s-direction, the cosine of this angle is $c(t) = \langle \dot\gamma(t), \frac{\partial}{\partial y}(\gamma(t))\rangle$. Because N is symmetric about σ, we can assume that $c(0) = 1/\sqrt{2}$, not $-1/\sqrt{2}$. Since γ is a geodesic and $\gamma(t) \in N$,

$$|c'(t)| = |\langle \dot\gamma(t), \nabla_{\dot\gamma(t)} \tfrac{\partial}{\partial y}(\gamma(t))\rangle| \le 0.11.$$

Suppose that $-1/2 \le t_0 \le t_1 \le 1/2$ and $\gamma(t)$ is defined for $t_0 \le t \le t_1$. If $t \in [t_0, t_1]$,

$$\cos(\pi/3) < \cos(\pi/4) - \tfrac{1}{2} \cdot 0.11 \le c(t) \le \cos(\pi/4) + \tfrac{1}{2} \cdot 0.11 < \cos(\pi/6).$$

We now show that γ hits ∂N before $t = 1/2$, in other words that $t_1 < 1/2$. For $0 \le t \le t_1$, we have $(y \circ \gamma)'(t) = c(t) > \cos(\pi/3)$. Thus if $t_1 = 1/2$,

$$y(\gamma(t_1)) > y(\gamma(0)) + \tfrac{1}{2}\cos(\pi/3) \ge -1/10 + 1/4 > 1/10,$$

which contradicts $\gamma(t_1) \in N$. Similarly $t_0 > -1/2$. □

3.3. **Definition.** *A geodesic in N is horizontal (resp. vertical) if the maximal angle that it makes with the s-direction (resp. y-direction) is at most $\pi/3$.*

It follows from Corollary 3.2 that every geodesic in N is horizontal or type y (or both). From Lemma 3.1, we easily obtain

3.4. **Corollary.** *Let $\gamma(t)$ be a unit speed geodesic in N.*
(i) *If γ is vertical, then $|(y \circ \gamma)'(t)| \geq 1/2$.*
(ii) *If γ is horizontal, then $|(s \circ \gamma)'(t)| > 1/3$.*
(iii) *If γ is horizontal, then $|(y \circ \gamma)'(t)| < 2|(s \circ \gamma)'(t)|$.*

4. Proof that S has no conjugate points

Proposition 4.4 below will show that along every geodesic γ in S the Riccati equation

$$u'(t) + u^2(t) + K(t) = 0, \tag{4.1}$$

where $K(t)$ is the curvature of S at $\gamma(t)$, has a solution $u_\gamma(t)$ that is defined for all $t \geq 0$. It then follows that S has no conjugate points. To see this, suppose that $J(t)$ is a Jacobi field along γ with $J(a) = 0 = J(b)$ and $J(t) \neq 0$ for $a < t < b$. Then $J(t) = j(t)N(t)$, where $N(t)$ is a continuous field of unit normals along γ and $j(t)$ is a solution of the scalar Jacobi equation $j''(t) + K(t)j(t) = 0$. It follows easily that $u = j'/j$ is a continuous solution of (4.1) on the interval (a, b) with $\lim_{t \searrow a} u(t) = \infty$ and $\lim_{t \nearrow b} u(t) = -\infty$, which is impossible since the graph of u would have to cross the graph of u_γ. For a fuller discussion, see [BBB, §1].

4.1. **Lemma.** *Let $u_i(t)$, $i = 0, 1$, be the solutions of the initial value problems*

$$u_i' + u^2 + K_i(t) = 0, \qquad u_i(0) = w_i, \qquad i = 0, 1.$$

Suppose $w_1 \geq w_0$, $K_1(t) \leq K_0(t)$ for $t \in [0, t_0]$, and $u_0(t_0)$ is defined. Then $u_1(t) \geq u_0(t)$ for $t \in [0, t_0]$.

Proof. The difference $\Delta u(t) = u_1(t) - u_0(t)$ satisfies the linear equation

$$\Delta u' = -(u_0 + u_1)\Delta u + K_0(t) - K_1(t). \quad \square$$

4.2. **Lemma.** (i) *If $u' + u^2 + 10^{-6} = 0$ and $u(t_0) = 0$, then*

$$u(t) = -10^{-3} \tan 10^{-3}(t - t_0).$$

(ii) *If $u' + u^2 - 1 = 0$ and $u(t_0) = -10^{-3}$, then*

$$u(t) = \tanh(t - t_0 - \operatorname{arctanh}(10^{-3})).$$

4.3. **Lemma.** *Let $\gamma(t)$ be a geodesic of the torus T defined in* §2. *Suppose that γ crosses one of the parallels of latitude $\{e^{i\alpha}\} \times_f S^1$ or $\{e^{-i\alpha}\} \times_f S^1$ transversally. Then there is a solution of the Riccati equation (4.1) along γ that is defined for all t and vanishes whenever $\gamma(t) \in [e^{i\alpha}, e^{-i\alpha}] \times_f S^1$.*

Innami [I, Lemma 1.1] has proved essentially the same lemma.

Proof. Let V be the Killing field defined by the rotational symmetry. Then V is tangent to the parallels of latitude and has length $f(s)$ on the parallel

$\{s\} \times_f S^1$. The Clairaut integral $\langle \dot\gamma(t), V(\gamma(t))\rangle$ is constant. Since $\|V\|$ attains its minimum on $\{e^{i\alpha}\} \times_f S^1$ and $\{e^{-i\alpha}\} \times_f S^1$ and γ is transverse to one of these curves, we have $\langle \dot\gamma(t), V(\gamma(t))\rangle < \|V(\gamma(t))\|$ for all t. Hence γ is never tangent to V. The restriction of V to γ is a Jacobi field. Let $j(t)$ be the component of $V(\gamma(t))$ orthogonal to γ. Then $u(t) = j'(t)/j(t)$ is a solution of the Riccati equation (4.1) along γ which is defined for all t. Since f is constant on $[e^{i\alpha}, e^{-i\alpha}]$, it follows that $j(t)$ is constant and $u(t) = 0$ when $\gamma(t) \in [e^{i\alpha}, e^{-i\alpha}] \times_f S^1$. □

Now consider our surface S. Let $R = R_- \cup R_0 \cup R_+ \subseteq N$, where

$$\begin{aligned} R_- &= [e^{-i\alpha}, e^{i\alpha}] \times [-10\alpha, -9\alpha], \\ R_0 &= [e^{-i\alpha}, e^{i\alpha}] \times [-9\alpha, 9\alpha], \\ R_+ &= [e^{-i\alpha}, e^{i\alpha}] \times [9\alpha, 10\alpha]. \end{aligned}$$

See Figure 1. The curvature of S is nonpositive outside R, by (2.2). For a geodesic γ of S define inductively a (possibly finite) sequence of times $t_0, t_1, \dots$ as follows. Choose t_0 so that $t_0 \le 0$ and $\gamma(t_0)$ is not in R (it is impossible to have $\gamma(t) \in R$ for all $t \le 0$, cf. Corollary 3.4). For $i \ge 1$, let t_i be the first time after t_{i-1} when $\gamma(t)$ enters R; if $\gamma(t)$ does not enter R for $t > t_{i-1}$, we set $t_i = \infty$ and the sequence terminates.

4.4. **Proposition.** *Let u_γ be the solution of* (4.1) *with $u_\gamma(t_0) = 0$. Suppose t_i is finite, $u_\gamma(t)$ is defined for $t_0 \le t \le t_i$ and $u_\gamma(t_i) \ge 0$. Then $u_\gamma(t)$ is defined for $t_i \le t < t_{i+1}$. If t_{i+1} is finite, then $u_\gamma(t_{i+1})$ is defined and $u_\gamma(t_{i+1}) \ge 0$.*

Proof. Since $u'_\gamma(t) < 0$ if $|u_\gamma|$ is large, $u_\gamma(t)$ can fail to be defined for all $t \ge t_0$ only if $u_\gamma(t) \to -\infty$ in finite time. Thus in order to show that $u_\gamma(t)$ is defined for $t_i \le t < t_{i+1}$, it suffices to bound $u_\gamma(t)$ from below.

Case 0. $i = 0$. For $t_0 \le t < t_1$, we have $K(t) \le 0$, since $\gamma(t)$ is not in R. Using Lemma 4.1 to compare $u_\gamma(t)$ with the solution of the initial value problem, $u' + u^2 = 0$ and $u(t_0) = 0$, shows that $u_\gamma(t) \ge 0$ for $t_0 \le t < t_1$. Moreover $u_\gamma(t_1) \ge 0$ if t_1 is finite.

When $i \ge 1$, let T_i be the time in (t_i, t_{i+1}) when γ leaves R. Since γ does not enter R for $T_i \le t < t_{i+1}$, the argument of Case 0 shows that it is enough to find a time $c_i \in [T_i, t_{i+1})$ such that $u_\gamma(t)$ is defined for $t_i \le t \le c_i$ and $u_\gamma(c_i) \ge 0$. We study three cases, depending on how γ crosses R while $t_i \le t < t_{i+1}$. See Figure 1. Set $y(t) = y(\gamma(t))$ and $s(t) = s(\gamma(t))$.

Case 1. γ is vertical. Clearly γ leaves $N = S^1 \times [-1/10, 1/10]$ while t is between t_i and t_{i+1}. Let b_i be the smallest time such that $b_i > t_i$ and $|y(b_i)| = 10\alpha$. By Corollary 3.4, $b_i - t_i \le 2|y(b_i) - y(t_i)| \le 40\alpha$. Recall from (2.1) that $K(t) \le 10^{-6}$. Since $u_\gamma(t_i) \ge 0$, we see from Lemmas 4.1 and 4.2(i) that, for $t_i \le t \le b_i$,

$$u_\gamma(t) \ge -10^{-3} \tan 10^{-3}(t - t_i) \ge -10^{-3} \tan(10^{-3} \cdot 40\alpha) \ge -10^{-3}.$$

Let c_i be the first time after b_i such that $\gamma(c_i) \in \partial N$. Then $c_i - b_i \ge 1/10 - 10\alpha = \alpha$, since $\alpha = 1/110$. By (2.3), $K(t) = -1$ for $b_i \le t \le c_i$. Since $u_\gamma(b_i) \ge -10^{-3}$, Lemmas 4.1 and 4.2(ii) show that

$$u_\gamma(t) \ge \tanh(t - b_i - \operatorname{arctanh}(10^{-3})) \quad \text{for } b_i \le t \le c_i.$$

Thus $u_\gamma(t)$ is defined for $t_i \le t \le c_i$ and $u_\gamma(c_i) \ge \tanh(\alpha - \operatorname{arctanh}(10^{-3})) \ge 0$.

Case 2. γ is horizontal and does not meet $R_- \cup R_+$. Let c_i be the time in (t_i, t_{i+1}) when γ leaves R. We can consider $\gamma|[t_i, c_i]$ as a geodesic segment in the torus T that stretches between the parallels of latitude $\{e^{-i\alpha}\} \times_f S^1$ and $\{e^{i\alpha}\} \times_f S^1$. Lemma 4.3 shows that there is a solution $U(t)$ of the Riccati equation along this segment that vanishes at the endpoints. Clearly $u_\gamma(t) \geq U(t)$ for $t_i \leq t \leq c_i$ and $u_\gamma(c_i) \geq 0$.

Case 3. γ is horizontal and enters $R_- \cup R_+$. Since N is symmetric about the curves $y = 0$ and $s = 1$, we may assume that γ enters R_+ and $s(t)$ moves anticlockwise on S^1 as t increases. Let b_i be the first time after t_i such that $s(t) = e^{2i\alpha}$. It is clear from Corollary 3.4(ii) that $b_i - t_i \leq 3\,\mathrm{dist}_{S^1}(s(b_i), s(t_i)) \leq 9\alpha$. As in Case 1, Lemmas 4.1 and 4.2(i) show that, for $t_i \leq t \leq b_i$,

$$u_\gamma(t) \geq -10^{-3} \tan 10^{-3}(t - t_i) \geq -10^{-3} \tan(10^{-3} \cdot 9\alpha) \geq -10^{-3}.$$

Choose $a_i \in [t_i, b_i]$ so that $\gamma(a_i) \in R_+$. Then $s(a_i) \in [e^{-i\alpha}, e^{i\alpha}]$ and, by Corollary 3.4(iii), $|y(b_i) - y(a_i)| \leq 2\,\mathrm{dist}_{S^1}(s(b_i), s(a_i)) \leq 6\alpha$. Since $y(a_i) \geq 9\alpha$, it follows that $y(b_i) \geq 3\alpha$.

Let $c_i = b_i + \alpha$. Then $y(t) \geq 2\alpha$ for $b_i \leq t \leq c_i$, when we also have $s(t) \in [e^{2i\alpha}, e^{3i\alpha}]$. It follows from (2.3) that $K(t) = -1$ for $b_i \leq t \leq c_i$. Since $u_\gamma(b_i) \geq -10^{-3}$, Lemmas 4.1 and 4.2(ii) show that

$$u_\gamma(t) \geq \tanh(t - b_i - \operatorname{arctanh}(10^{-3})) \quad \text{for } b_i \leq t \leq c_i.$$

Thus $u_\gamma(t)$ is defined for $t_i \leq t \leq c_i$ and $u_\gamma(c_i) \geq \tanh(\alpha - \operatorname{arctanh}(10^{-3})) \geq 0$. $\square$

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