THE FLAT STRIP THEOREM FAILS
FOR SURFACES WITH NO CONJUGATE POINTS

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Abstract. A compact \( C^\infty \) surface with no conjugate points is constructed so that there are two homotopic closed geodesics that do not bound a flat annulus.

One of the basic rigidity properties of manifolds with nonpositive curvature is the

**Flat Strip Theorem** [EON]. Let \( \beta \) and \( \gamma \) be geodesics in a simply connected manifold with nonpositive curvature. Suppose that \( \beta \) and \( \gamma \) have finite Hausdorff distance. Then \( \beta \) and \( \gamma \) are the edges of a flat strip, i.e., an isometrically and totally geodesically embedded copy of \( I \times \mathbb{R} \).

Two natural generalizations of nonpositive curvature are the no focal point and no conjugate point properties. These can be described in terms of Jacobi fields [OS1, Proposition 4]. A complete Riemannian manifold has

(i) nonpositive curvature if and only if \( \langle Y, Y \rangle'' \geq 0 \) for every Jacobi field \( Y \);
(ii) no focal points if and only if \( \langle Y, Y \rangle'(t) > 0 \) whenever \( t > 0 \) and \( Y \) is a nontrivial Jacobi field with \( Y(0) = 0 \);
(iii) no conjugate points if and only if \( \langle Y, Y \rangle(t) > 0 \) whenever \( t > 0 \) and \( Y \) is a nontrivial Jacobi field with \( Y(0) = 0 \).

Gulliver [G] has given examples that show (iii) \( \not\Rightarrow \) (ii) \( \not\Rightarrow \) (i).

The flat strip theorem holds for manifolds with no focal points [OS2, E]. The present paper shows that it fails for surfaces with no conjugate points. We construct a compact \( C^\infty \) surface \( S \) with no conjugate points that contains a closed annulus that is foliated by homotopic closed geodesics, but is not flat. A lift of this annulus to the universal cover is a nonflat strip bounded by two geodesics with finite Hausdorff distance.

Here is a brief description of \( S \). Consider a torus of revolution \( T \) that is flat except for one small bulge. The torus \( T \) is foliated by closed meridian

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geodesics. Let $\Sigma_1$ be a thin annulus that is a union of meridians. A $C^1$ approximation to $S$ can be constructed by taking a surface with constant curvature $-1$ that contains a simple closed geodesic with the same length as the meridians of $T$, cutting the surface along this geodesic and then gluing in $\Sigma_1$. One could think of $S$ as being obtained from this surface by carefully smoothing the metric near $\partial \Sigma_1$. Instead we construct a $C^\infty$ Riemannian metric $g$ on $S^1 \times \mathbb{R}$ so that

(i) the curve $\sigma = S^1 \times \{0\}$ is a geodesic;
(ii) the natural coordinates $s \in S^1$ and $y \in \mathbb{R}$ are Fermi coordinates for $\sigma$;
(iii) $\sigma$ has a neighbourhood $\Sigma$ isometric to $\Sigma_1$;
(iv) the curvature is $-1$ outside a neighbourhood of $\Sigma$.

The two ends of $(S^1 \times \mathbb{R}, g)$ are then replaced by compact ends with curvature $-1$, using a standard construction. Property (ii) of $g$ makes it easy to prescribe the curvature near $\partial \Sigma$. Note finally that $S$ has focal points because there are periodic Jacobi fields along $\sigma$.

1. Preliminaries

Suppose we define a metric on a surface with respect to the coordinates $s$ and $y$ by

$$
\begin{pmatrix}
g_{ss} & g_{sy} \\
g_{ys} & g_{yy}
\end{pmatrix} = \begin{pmatrix} \psi^2(s, y) & 0 \\ 0 & 1 \end{pmatrix},
$$

where $\psi > 0$ and $\psi$ is the solution of the initial value problem

$$
(1.2) \quad \psi_{yy}(s, y) + K(s, y)\psi(s, y) = 0, \quad \psi(s, 0) = 1, \quad \psi_y(s, 0) = 0.
$$

Then the curve $\sigma$ defined by $y = 0$ is a geodesic for which $s$ and $y$ are Fermi coordinates; $s$ measures arclength along $\sigma$ and $|y|$ is distance from $\sigma$. The curvature at the point with coordinates $(s, y)$ is $K(s, y)$. In our case, $\sigma$ will be a closed geodesic of length $2\pi$, so we take $s \in S^1$. If $s_1, s_2 \in S^1$, $(s_1, s_2)$ and $[s_1, s_2]$ will denote the open and closed arcs respectively from $s_1$ to $s_2$ in the anticlockwise direction.

2. Construction of $T$ and $S$

Let $\alpha = 1/110$. The torus $T$ will be a warped product $S^1 \times_f S^1$ (warped products are described in [BON]), where $f$ is a $C^\infty$ function on $S^1$ such that $f(s) = 1$ if $s \in [e^{i\alpha}, e^{-i\alpha}]$ and $1 < f(s) < 9$ if $s \in (e^{-i\alpha}, e^{i\alpha})$. Thus $T$ is a torus of revolution with the curves $S^1 \times_f \{s\}, s \in S^1$, as meridian geodesics. Let $\sigma_1$ be the geodesic $S^1 \times_f \{1\}$. We choose $f$ so that the curvature $K_T$ of $T$ satisfies $|K_T| \leq 10^{-6}$ throughout $T$. It is clear from this that there is a diffeomorphism $\pi : S^1 \times [-1/10, 1/10] \to \{p \in T : \text{dist}(p, \sigma_1) \leq 1/10\}$ such that $(s, y)$ are the Fermi coordinates with respect to $\sigma_1$ for the point $\pi(s, y)$.

We now choose the curvature $K(s, y)$ that we want for the metric $g$ on $S^1 \times \mathbb{R}$. Choose a $C^\infty$ function $h : S^1 \to [\alpha, 9\alpha]$ such that

$$h(s) = \begin{cases} 
9\alpha & \text{if } s \in [e^{-i\alpha}, e^{i\alpha}], \\
\alpha & \text{if } s \in [e^{2i\alpha}, e^{-2i\alpha}].
\end{cases}$$
Let \( H = \{ (s, y) \in S^1 \times \mathbb{R} : \|y\| \leq h(s) \} \) and \( H_\alpha = \{ (s, y) \in S^1 \times \mathbb{R} : \|y\| \leq h(s) + \alpha \} \). Choose a \( C^\infty \) function \( \theta : S^1 \times \mathbb{R} \to [0, 1] \) with \( \theta = 1 \) on \( H \) and \( \theta = 0 \) outside \( H_\alpha \). Define \( K(s, y) \) by

\[
K(s, y) = \begin{cases} 
K_T(\pi(s, y)) & \text{if } (s, y) \in H, \\
\theta(s, y)K_T(\pi(s, y)) - (1 - \theta(s, y)) & \text{if } (s, y) \in H_\alpha \setminus H, \\
-1 & \text{if } (s, y) \notin H_\alpha.
\end{cases}
\]

See Figure 1. Observe that \( K_T \) vanishes in the region \([e^{ia}, e^{-ia}] \times \mathbb{R} \) and

\begin{align*}
(2.1) & \quad -1 \leq K \leq 10^{-6} \text{ everywhere;} \\
(2.2) & \quad K \leq 0 \text{ outside } (e^{-ia}, e^{ia}) \times (-10\alpha, 10\alpha); \\
(2.3) & \quad K = -1 \text{ outside } ((e^{-2ia}, e^{2ia}) \times (-10\alpha, 10\alpha)) \cup (S^1 \times (-2\alpha, 2\alpha)).
\end{align*}

Finally \( g \) is the metric represented with respect to the \( s-y \) coordinates by the matrix (1.1) in which \( \psi \) satisfies (1.2). It is easily shown using (2.1) and

![Figure 1](image-url)
that \( \psi(s, y) > 0 \) for all \((s, y)\). One can even see that \( \psi(s, y) \to \infty \) uniformly as \(|y| \to \infty \). The construction ensures that the curve \( \sigma(t) = (e^{it}, 0) \) is a unit speed geodesic for the metric \( g \) and \((s, y)\) are Fermi coordinates for \( \sigma \) throughout \( S^1 \times \mathbb{R} \).

Let \( \Sigma_1 = S^1 \times_f [e^{-ia}, e^{ia}] \), so \( \Sigma_1 \) is a neighbourhood of \( \sigma_1 \) foliated by meridian geodesics. It is clear from the choice of \( f \) and \( g \) that \( \Sigma_1 \subseteq \pi(H) \).

Since \( K = K_T \circ \pi \) on \( H \), we see that \( \Sigma \defeq \pi^{-1}(\Sigma_1) \) is isometric to \( \Sigma_1 \).

We now outline the procedure for replacing the ends of \((S^1 \times \mathbb{R}, g)\) with compact ends. It follows a similar construction in [BBB]. If we cut a component of \((S^1 \times \mathbb{R}) \setminus N\) along the geodesic \( \{1\} \times \mathbb{R} \) we obtain a fan-shaped subset of the Poincaré disc shown in Figure 2.

This subset is bounded by two geodesic rays \( \eta \) and \( \eta' \) and a curve corresponding to a component of \( \partial N \). For any large enough \( n \), it is possible to draw a sequence of \( 4n + 1 \) hyperbolic geodesic segments \( d'_n, c'_n, d'_{n-1}, \ldots, d'_1, c'_1, d_0, c''_n, d''_n, \ldots, c''_1, d''_0 \) such that

- \( d'_n \) and \( d''_n \) are orthogonal to \( \eta' \) and \( \eta'' \) respectively;
- adjacent geodesics in the sequence are orthogonal;
- \( c'_i \) and \( c''_i \) have the same length for \( 1 \leq i \leq n \).

By identifying \( \eta' \) with \( \eta'' \) and \( c'_i \) with \( c''_i \), \( 1 \leq i \leq n \), we obtain a hyperbolic metric on the sphere with \( n + 2 \) punctures. One of the holes is bounded by a copy of a component of \( \partial N \) and the others by closed geodesics \( d_0, \ldots, d_n \). Now we can adjoin handles with curvature \(-1\) along \( d_0, \ldots, d_n \).
3. Geodesics in $N$

Let $N = S^1 \times [-1/10, 1/10]$. Note that $H_\alpha \subseteq N$, since $10\alpha < 1/10$.

3.1. Lemma. For any $(s, y) \in N$ and any unit vector $u \in T_{(s, y)}N$,

(i) $0.99 \leq \|\frac{\partial}{\partial s}(s, y)\| \leq 1.01$;

(ii) $\|\nabla u \frac{\partial}{\partial y}\| \leq 0.11$.

Proof. Note that $|K(s, y)| \leq 1$, by (2.1). Since $|y| \leq 1/10$, it follows from (1.2) and the Sturm Comparison Theorem that

$$0.99 \leq \cos(1/10) \leq \psi(s, y) \leq \cosh(1/10) \leq 1.01.$$ 

This proves (i), since $\psi = \|\frac{\partial}{\partial s}\|$.

Since the level curves for $s$ are the geodesics orthogonal to $\sigma$ and $\frac{\partial}{\partial y}(s, 0)$ is a smooth field of unit vectors normal to $\sigma$,

$$\nabla \frac{\partial}{\partial y}(s, y) = 0 \quad \text{and} \quad \nabla \frac{\partial}{\partial y}(s, 0) = \nabla \frac{\partial}{\partial y}(s, 0) = 0.$$ 

Since $\frac{\partial}{\partial s}$ is a Jacobi field along each geodesic $s = \text{const}$,

$$\nabla \frac{\partial}{\partial y} \frac{\partial}{\partial s} \nabla \frac{\partial}{\partial y} = \nabla \frac{\partial}{\partial y} \nabla \frac{\partial}{\partial s} = -R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y},$$ 

where $R$ is the curvature tensor for $g$. Thus if $u = a \frac{\partial}{\partial y}(s, y) + b \frac{\partial}{\partial y}(s, y)$, we have

$$\|\nabla u \frac{\partial}{\partial y}\| = \|a \nabla\frac{\partial}{\partial y}\frac{\partial}{\partial s}\| \leq |a| \int_0^y \|R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y})(\frac{\partial}{\partial s}, \frac{\partial}{\partial y})\| dz.$$ 

Since $(R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}) = 0$ and $\|\frac{\partial}{\partial y}\| = 1$,

$$\|R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}\| = \|\frac{\partial}{\partial s}\|^{-1} \|R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}\| = 0.$$ 

Thus $\|\nabla u \frac{\partial}{\partial y}\| \leq |a| \cdot |y| \cdot \sup_N \|\frac{\partial}{\partial s}(s, y)\| \leq (0.99)^{-1} \cdot (1/10) \cdot 1.01 \leq 0.11$, by (i). □

3.2. Corollary. Let $\gamma$ be a maximal unit speed geodesic in $N$ that makes angle $\pi/4$ with the $s$ and $y$ directions at $\gamma(0)$. Then the angle between $\gamma$ and the $s$ direction is always between $\pi/6$ and $\pi/3$.

Proof. Since $\frac{\partial}{\partial y}$ is the unit vector field in the $s$-direction, the cosine of this angle is $c(t) = \langle \gamma'(t), \frac{\partial}{\partial y}(\gamma(t)) \rangle$. Because $N$ is symmetric about $\sigma$, we can assume that $c(0) = 1/\sqrt{2}$, not $-1/\sqrt{2}$. Since $\gamma$ is a geodesic and $\gamma(t) \in N$,

$$|c'(t)| = |\langle \gamma'(t), \nabla_{\gamma(t)} \frac{\partial}{\partial y}(\gamma(t)) \rangle| \leq 0.11.$$ 

Suppose that $-1/2 \leq t_0 \leq t_1 \leq 1/2$ and $\gamma(t)$ is defined for $t_0 \leq t \leq t_1$. If $t \in [t_0, t_1]$,

$$\cos(\pi/3) < \cos(\pi/4) - 0.11 \leq c(t) \leq \cos(\pi/4) + 0.11 < \cos(\pi/6).$$ 

We now show that $\gamma$ hits $\partial N$ before $t = 1/2$, in other words that $t_1 < 1/2$. For $0 \leq t \leq t_1$, we have $(\gamma \circ \gamma)'(t) = c(t) > \cos(\pi/3)$. Thus if $t_1 = 1/2$,

$$y(\gamma(t_1)) > y(\gamma(0)) + \frac{1}{2} \cos(\pi/3) \geq -1/10 + 1/4 > 1/10,
3.3. Definition. A geodesic in $N$ is horizontal (resp. vertical) if the maximal angle that it makes with the $s$-direction (resp. $y$-direction) is at most $\pi/3$.

It follows from Corollary 3.2 that every geodesic in $N$ is horizontal or type $y$ (or both). From Lemma 3.1, we easily obtain

3.4. Corollary. Let $y(t)$ be a unit speed geodesic in $N$.

(i) If $y$ is vertical, then $|(y \circ y)'(t)| \geq 1/2$.
(ii) If $y$ is horizontal, then $|(s \circ y)'(t)| > 1/3$.
(iii) If $y$ is horizontal, then $|(y \circ y)'(t)| < 2|(s \circ y)'(t)|$.

4. Proof that $S$ has no conjugate points

Proposition 4.4 below will show that along every geodesic $y$ in $S$ the Riccati equation

$$u'(t) + u^2(t) + K(t) = 0,$$

where $K(t)$ is the curvature of $S$ at $y(t)$, has a solution $u_y(t)$ that is defined for all $t \geq 0$. It then follows that $S$ has no conjugate points. To see this, suppose that $J(t)$ is a Jacobi field along $y$ with $J(a) = 0 = J(b)$ and $J(t) \neq 0$ for $a < t < b$. Then $J(t) = j(t)N(t)$, where $N(t)$ is a continuous field of unit normals along $y$ and $j(t)$ is a solution of the scalar Jacobi equation $j''(t) + K(t)j(t) = 0$. It follows easily that $u = j'/j$ is a continuous solution of (4.1) on the interval $(a, b)$ with $\lim_{t \to a} u(t) = \infty$ and $\lim_{t \to b} u(t) = -\infty$, which is impossible since the graph of $u$ would have to cross the graph of $u_y$. For a fuller discussion, see [BBB, §1].

4.1. Lemma. Let $u_i(t), i = 0, 1,$ be the solutions of the initial value problems

$$u_i'(t) + u_i^2(t) + K_i(t) = 0, \quad u_i(0) = w_i, \quad i = 0, 1.$$

Suppose $w_1 \geq w_0, K_1(t) \leq K_0(t)$ for $t \in [0, t_0]$, and $u_0(t_0)$ is defined. Then $u_1(t) \geq u_0(t_0)$ for $t \in [0, t_0]$.

Proof. The difference $\Delta u(t) = u_1(t) - u_0(t)$ satisfies the linear equation

$$\Delta u' = -(u_0 + u_1)\Delta u + K_0(t) - K_1(t).$$

4.2. Lemma. (i) If $u' + u^2 + 10^{-6} = 0$ and $u(t_0) = 0$, then

$$u(t) = -10^{-3}\tan 10^{-3}(t - t_0).$$

(ii) If $u' + u^2 + 1 = 0$ and $u(t_0) = -10^{-3}$, then

$$u(t) = \tanh(t - t_0 - \arctanh(10^{-3})).$$

4.3. Lemma. Let $y(t)$ be a geodesic of the torus $T$ defined in §2. Suppose that $y$ crosses one of the parallels of latitude $\{e^{i\alpha}\} \times_f S^1$ or $\{e^{-i\alpha}\} \times_f S^1$ transversally. Then there is a solution of the Riccati equation (4.1) along $y$ that is defined for all $t$ and vanishes whenever $y(t) \in [e^{i\alpha}, e^{-i\alpha}] \times_f S^1$.

Innami [I, Lemma 1.1] has proved essentially the same lemma.

Proof. Let $V$ be the Killing field defined by the rotational symmetry. Then $V$ is tangent to the parallels of latitude and has length $f(s)$ on the parallel...
The Clairaut integral \( \langle \dot{y}(t), V(y(t)) \rangle \) is constant. Since \( \|V\| \) attains its minimum on \( \{e^{ia}\} \times_f S^1 \) and \( \{e^{-ia}\} \times_f S^1 \) and \( y \) is transverse to one of these curves, we have \( \langle \dot{y}(t), V(y(t)) \rangle < \|V(y(t))\| \) for all \( t \). Hence \( y \) is never tangent to \( V \). The restriction of \( V \) to \( y \) is a Jacobi field. Let \( j(t) \) be the component of \( V(y(t)) \) orthogonal to \( y \). Then \( u(t) = f'(t)/j(t) \) is a solution of the Riccati equation (4.1) along \( y \) which is defined for all \( t \). Since \( f \) is constant on \( [e^{ia}, e^{-ia}] \), it follows that \( j(t) \) is constant and \( u(0) = 0 \) when \( y(t) \in [e^{ia}, e^{-ia}] \times_f S^1 \).

Now consider our surface \( S \). Let \( R = R_- \cup R_0 \cup R_+ \subseteq N \), where
\[
R_- = [e^{-ia}, e^{ia}] \times [-10\alpha, -9\alpha],
R_0 = [e^{-ia}, e^{ia}] \times [-9\alpha, 9\alpha],
R_+ = [e^{-ia}, e^{ia}] \times [9\alpha, 10\alpha].
\]
See Figure 1. The curvature of \( S \) is nonpositive outside \( R \), by (2.2). For a geodesic \( y \) of \( S \) define inductively a (possibly finite) sequence of times \( t_0, t_1, \ldots \) as follows. Choose \( t_0 \) so that \( t_0 \leq 0 \) and \( \gamma(t_0) \) is not in \( R \) (it is impossible to have \( \gamma(t) \in R \) for all \( t \leq 0 \), cf. Corollary 3.4). For \( i \geq 1 \), let \( t_i \) be the first time after \( t_{i-1} \) when \( \gamma(t) \) enters \( R \); if \( \gamma(t) \) does not enter \( R \) for \( t > t_i \), we set \( t_i = \infty \) and the sequence terminates.

4.4. Proposition. Let \( u_y \) be the solution of (4.1) with \( u_y(t_0) = 0 \). Suppose \( t_i \) is finite, \( u_y(t) \) is defined for \( t_0 \leq t \leq t_i \) and \( u_y(t_i) \geq 0 \). Then \( u_y(t) \) is defined for \( t_i < t < t_{i+1} \). If \( t_{i+1} \) is finite, then \( u_y(t_{i+1}) \) is defined and \( u_y(t_{i+1}) \geq 0 \).

Proof. Since \( u_y'(t) < 0 \) if \( |u_y| \) is large, \( u_y(t) \) can fail to be defined for all \( t \geq t_0 \) only if \( u_y(t) \to -\infty \) in finite time. Thus in order to show that \( u_y(t) \) is defined for \( t_i < t < t_{i+1} \), it suffices to bound \( u_y(t) \) from below.

Case 0. \( i = 0 \). For \( t_0 \leq t < t_1 \), we have \( K(t) \leq 0 \), since \( \gamma(t) \) is not in \( R \). Using Lemma 4.1 to compare \( u_y(t) \) with the solution of the initial value problem, \( u' + u^2 = 0 \) and \( u(t_0) = 0 \), shows that \( u_y(t) \geq 0 \) for \( t_0 \leq t < t_1 \). Moreover \( u_y(t_1) \geq 0 \) if \( t_1 \) is finite.

When \( t \geq 1 \), let \( T_i \) be the time in \( (t_i, t_{i+1}) \) when \( \gamma \) leaves \( R \). Since \( \gamma \) does not enter \( R \) for \( t_i < t < t_{i+1} \), the argument of Case 0 shows that it is enough to find a time \( c_i \in (T_i, t_{i+1}) \) such that \( u_y(t) \) is defined for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq 0 \). We study three cases, depending on how \( \gamma \) crosses \( R \) while \( t_i < t < t_{i+1} \). See Figure 1. Set \( y(t) = y(\gamma(t)) \) and \( s(t) = s(\gamma(t)) \).

Case 1. \( \gamma \) is vertical. Clearly \( \gamma \) leaves \( N = S^1 \times [-1/10, 1/10] \) while \( t \) is between \( t_i \) and \( t_{i+1} \). Let \( b_i \) be the smallest time such that \( b_i > t_i \) and \( |y(b_i)| = 10\alpha \). By Corollary 3.4, \( b_i - t_i \leq 2|y(b_i) - y(t_i)| \leq 40\alpha \). Recall from (2.1) that \( K(t) \leq 10^{-6} \). Since \( u_y(t_i) \geq 0 \), we see from Lemmas 4.1 and 4.2(i) that, for \( t_i \leq t \leq b_i \),
\[
u_y(t) \geq -10^{-3} \tan 10^{-3}(t - t_i) \geq -10^{-3} \tan(10^{-3} \cdot 40\alpha) \geq -10^{-3}.
\]
Let \( c_i \) be the first time after \( b_i \) such that \( \gamma(c_i) \in \partial N \). Then \( c_i - b_i \geq 1/10 - 10\alpha = \alpha \), since \( \alpha = 1/110 \). By (2.3), \( K(t) = -1 \) for \( b_i \leq t \leq c_i \). Since \( u_y(b_i) \geq -10^{-3} \), Lemmas 4.1 and 4.2(ii) show that
\[
u_y(t) \geq \tanh(t - b_i - \alpha - \tanh(10^{-3})) \quad \text{for} \quad b_i \leq t \leq c_i.
\]
Thus \( u_y(t) \) is defined for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq \tanh(\alpha - \alpha - \tanh(10^{-3})) \geq 0 \).
Case 2. \( y \) is horizontal and does not meet \( R_- \cup R_+ \). Let \( c_i \) be the time in \((t_i, t_{i+1})\) when \( y \) leaves \( R \). We can consider \( y|[t_i, c_i] \) as a geodesic segment in the torus \( T \) that stretches between the parallels of latitude \( \{e^{-ia}\} \times S^1 \) and \( \{e^{ia}\} \times S^1 \). Lemma 4.3 shows that there is a solution \( U(t) \) of the Riccati equation along this segment that vanishes at the endpoints. Clearly \( u_y(t) \geq U(t) \) for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq 0 \).

Case 3. \( y \) is horizontal and enters \( R_- \cup R_+ \). Since \( N \) is symmetric about the curves \( y = 0 \) and \( s = 1 \), we may assume that \( y \) enters \( R_+ \) and \( s(t) \) moves anticlockwise on \( S^1 \) as \( t \) increases. Let \( b_i \) be the first time after \( t_i \) such that \( s(t) = e^{2ia} \). It is clear from Corollary 3.4(ii) that \( b_i - t_i \leq 3 \text{dist}_{S^1}(s(b_i), s(t_i)) \leq 9\alpha \). As in Case 1, Lemmas 4.1 and 4.2(ii) show that, for \( t_i < t < b_i \),

\[
u_y(t) \geq -10^{-3} \tan 10^{-3}(t - t_i) \geq -10^{-3} \tan(10^{-3} \cdot 9\alpha) \geq -10^{-3}.
\]

Choose \( a_i \in [t_i, b_i] \) so that \( y(a_i) \in R_+ \). Then \( s(a_i) \in [e^{-ia}, e^{ia}] \) and, by Corollary 3.4(iii), \( |y(b_i) - y(a_i)| \leq 2 \text{dist}_{S^1}(s(b_i), s(a_i)) \leq 6\alpha \). Since \( y(a_i) \geq 9\alpha \), it follows that \( y(b_i) \geq 3\alpha \).

Let \( c_i = b_i + \alpha \). Then \( y(t) \geq 2\alpha \) for \( b_i < t \leq c_i \), when we also have \( s(t) \in [e^{2ia}, e^{3ia}] \). It follows from (2.3) that \( K(t) = -1 \) for \( b_i \leq t \leq c_i \). Since \( u_y(b_i) \geq -10^{-3} \), Lemmas 4.1 and 4.2(ii) show that

\[
u_y(t) \geq \tanh(t - b_i - \arctanh(10^{-3})) \text{ for } b_i \leq t \leq c_i.
\]

Thus \( u_y(t) \) is defined for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq \tanh(\alpha - \arctanh(10^{-3})) \geq 0 \). \( \Box \)

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