REGULAR MATRICES AND P-SETS IN $\beta N \setminus N$. II

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ABSTRACT. It was discovered by Henriksen and Isbell that the support in $\beta N \setminus N$ of a regular matrix is a P-set. We study conditions under which a P-subset of a matrix support set contains another matrix support set.

1. Introduction

Let $T = (t_{mn})$ be a nonnegative regular matrix. By regularity, $T$ satisfies

(i) $\sup_n \sum_k t_{nk} < \infty$,
(ii) $\lim_{n \to \infty} \sum_k t_{nk} = 1$,
(iii) $\lim_{k \to \infty} t_{nk} = 0$ for all $n$.

$T$ defines a linear operator on the space $C_b(N)$ of all real-valued bounded functions on the positive integers $N$, by the formula $Tf(n) = \sum_b t_{nk} f(k)$. Classically, regular matrices were of interest because they served to extend the concept of limit: if $\lim_{n \to \infty} f(n) = c$ exists then $\lim_{n \to \infty} Tf(n) = c$, and this latter limit (called $T$-lim$(f)$) may exist for many functions that do not have a limit in the usual sense.

If we restrict attention to functions of the form $1_A$, where $A \subset N$, then we are able to make an interesting connection with the topology of $N^* = \beta N \setminus N$. Let $F_T$ be the filter of subsets of $N$ such that $T$-lim $1_A = 1$, and let $K_T$ be the corresponding closed set in $N^*$, to be defined in the next section. $K_T$ is called the support of $T$. In 1964 Henriksen and Isbell [HI] showed that $K_T$ is a P-set, i.e., is interior to any closed $G_\delta$ set that contains it. (For the special case of the Cesaro matrix, this is even implicit in [H, p. 38].) In [A1] it is shown that, under the assumption $N^*$ contains a dense set of $P$-points, every $K_T$ set contains a family of $2^c$ pairwise disjoint P-sets, each the support of a regular Borel probability measure on $N^*$. (The support of a Borel probability measure is the intersection of all closed sets of measure 1.) Since a matrix support set cannot satisfy the countable chain condition [HI], it is clear that a P-set that is the support of a probability measure cannot be a matrix support set, so that under the continuum hypothesis ($C-H$), not every nowhere dense infinite P-set in $N^*$ is the support of a matrix. (However it is consistent with ZFC that there are no c.c.c. P-sets in $N^*$ [FSZ].) Just and Krawczyk [JK] showed

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that the supports of a large class of matrices are homeomorphic to each other, but not to $N^*$. More recently, Winfried Just has shown that no nowhere dense $K_T$ set is homeomorphic to $N^*$, and he has also produced (without set theoretic hypotheses) a nowhere dense P-set in $N^*$ that is not a $K_T$ set. There still remains the problem of characterizing those nowhere dense P-sets that are supports of matrices.

In this paper we make a first effort at finding P-sets that contain matrix supports. If $K_T$ is a support set and $L \subset K_T$ is a P-set, we give necessary and sufficient conditions for $L$ to contain the support set of some submatrix of $T$.

There remains the problem of when $L$ is exactly the support of a submatrix of $T$.

The general theory of P-sets in topological spaces has been developed by A. I. Veksler and his colleagues. (See, e.g., [V], and also [A2].) For a survey P-sets in $N^*$, see [M]. For compactification and summability see [FT].

2. Preliminaries

We repeat some definitions from [A1]. If $A \subset N$, let $A'$ be its closure in $\beta N$ and $A^* = A' \cap (\beta N \setminus N) = A' \cap N^*$. Then $K_T = \bigcap\{A^* : A \in F_T\}$. If $f \in C_b(N)$, the space of bounded real functions on $N$, then $f'$ is its extension to $\beta N$ and $f^* = f'|N^*$.

$T = (t_{mn})$ is a positive regular matrix. If $c_0$ is the space of real functions on $N$ vanishing at infinity then $T(c_0) \subset c_0$, so $T$ induces an operator $T^*$ on $C(N^*)$ by the formula $T^* f^* = (Tf)^*$. Also, $K_T = \bigcap\{A^* : T^{-1}\lim 1_A = 1\} = \bigcap\{A^* : A \in F_T\}$. We characterize $K_T$ another way. If $p \in N^*$, let $t_p$ be the Borel probability representing the functional $f \rightarrow T^* f(p)$ for $f \in C(N^*)$, so we have $T^* f(p) = \int f \, dt_p$. Then it is easy to see that $K_T = \text{closure } \{K_p : p \in N^*\}$, where $K_p$ is the support set of $t_p$. Below, $L$ will be a P-set in $N^*$ such that $L \subset K_T$. If $F' = \{A \subset N : L \subset A^*\}$ then $F_T \subset F'$.

If $f \in C_b(N)$, we write $T f(n) = \sum_k t_{nk} f(k)$. If $f = 1_A$, the indicator of $A$, we may write $T 1_A(n) = t_n(A)$.

A basic fact about $N^*$ is that a nonvoid open set cannot be written as a union of cardinality $c$ of closed nowhere dense sets [P, p. 46].

**Main Theorem.** The following are equivalent:

(a) there exists an infinite $A = \{n(k)\} \subset N$ such that for all $B \in F'$,

$$\liminf t_{n(k)}(B) > 0;$$

(b) there exists an infinite $A = \{n(k)\}$ and infinite $E \subset N$ with

$$\liminf t_{n(k)}(E) > 0,$$

such that $L$ contains the support in $N^*$ of the regular matrix operator

$$R f(k) = T(f 1_E(n(k)))/T 1_E(n(k)).$$

3. Proof

"(b) implies (a)." Let $F_R$ be the filter of sets summable to 1 by $R$, and let $K_R$ be the corresponding closed set in $N^*$. If $L \supset K_R$ then $F' \subset F_R$, so for
B ∈ F',

\[ 1 = \lim_{k \to \infty} \frac{t_n(k)(B \cap E)}{t_n(k)(E)} \leq \lim_{k \to \infty} \frac{t_n(k)(B)}{t_n(k)(E)}. \]

Since \( \lim \inf t_n(k)(E) > 0 \), it follows that \( \lim \inf t_n(k)(B) > 0 \), so (a) holds.

For "(a) implies (b)," we need some lemmas.

3.1. Lemma. The function \( t_p(L) \) defined for \( p \in N^* \) is upper semicontinuous, so that \( \{ p : t_p(L) \geq t \} \) is closed for all real \( t \).

Proof. By regularity of the measure \( t_p \),

\[ t_p(L) = \inf\{ t_p(A^*) : A \in F' \} \]

\[ = \inf\{ T1_{A^*}(p) : A \in F' \}. \]

Since each \( T1_{A^*} \) is continuous on \( C(N^*) \), the result follows.

3.2. Lemma. \( \{ p \in N^* : t_p(L) > 0 \} \) has nonvoid interior.

Proof. Let \( A \) be as in hypothesis (a). We will prove \( A^* \) is contained in the set in question. Suppose not. Then there exists \( p \in A^* \) such that \( t_p(L) = 0 \).

By regularity of \( t_p \), there exists for each \( n \) a \( B_n \subset N \) such that \( L \subset B_n \) and \( t_p(B_n^*) < 1/n \). Since \( L \) is a P-set, there exists \( B \in F' \) with \( L \subset B^* \subset B_n^* \) for all \( n \), so \( t_p(B^*) = 0 \). Since \( p \in A^* \) and \( A = (n(k)) \), we get \( \lim \inf t_n(k)(B) = 0 \), contrary to (a).

3.3. Remark. If \( L \subset K_T \), there always exists some \( p \) such that \( t_p(L) > 0 \).

(Proof. Suppose \( t_p(L) = 0 \) for all \( p \in N^* \). Since \( t_p(L) = \lim\{ t_p(A^*) : A \in F' \} \), a downward directed set of continuous functions, Dini’s theorem implies the convergence is uniform. Hence for any \( n \), there exists \( A_n \in F' \) with \( t_p(A_n^*) < 1/n \). Since \( F' \) is a P-filter, there exists \( A \in F' \) with \( A^* \subset A_n^* \) for all \( n \), whence \( t_p(A^*) = 0 \) for all \( p \). Since \( A^* \) is clopen, \( A^* \cap K_T = \emptyset \), whence \( L \cap K_T = \emptyset \), a contradiction.)

However there may not be “enough” \( p \) with \( t_p(L) > 0 \). For instance, if we let \( T \) be the identity matrix and \( L \) any P-set nowhere dense in \( N^* \), then it is easy to see that no submatrix of \( T \) can have its support contained in \( L \). Thus some hypothesis like (a) is really needed.

3.4. Lemma. For each \( p \in N^* \), there exists \( A \in F' \) such that \( t_p(A^*) = t_p(L) \).

Proof. \( t_p(L) = \inf\{ t_p(A^*) : A \in F' \} \), so for each \( n \), there exists \( A_n \in F' \) with \( t_p(A_n^*) < t_p(L) + 1/n \). If \( A \in F' \) and \( A^* \subset A_n^* \) for all \( n \), then \( t_p(L) \leq t_p(A^*) < t_p(L) + 1/n \) for all \( n \).

3.5. Lemma. Let \( C \) be a nonvoid clopen subset of \( \{ p : t_p(L) > 0 \} \). Then there exists \( s > 0 \) such that \( C \cap \{ p : t_p(L) \geq s \} \) has nonvoid interior.

Proof. Let \( R_k = \{ p : t_p(L) \geq 1/k \} \), which is closed by 3.1. Then \( C = \bigcup_k (R_k \cap C) \), so by Baire category, there exists \( k \) so that \( R_k \cap C \) has nonvoid interior.

3.6. Lemma. There exists nonvoid clopen \( A^* \subset N^* \) and \( E \in F' \) such that for all \( p \in A^* \), \( t_p(E^*) = t_p(L) \geq s \), where \( s \) is as in 3.5.

Proof. Let \( C \) and \( s \) be as in 3.5, and let \( W \) be a clopen subset of \( C \cap \{ p : t_p(L) \geq s \} \). By 3.4, for each \( p \in W \), there exists \( A_p \in F' \) with \( t_p(L) = t_p(A_p^*) \geq s \). Let

\[ Z_p = \{ q \in W : t_q(A_p^*) = t_q(L) \}, \]
a closed set. (Note that \( t_q(L) - t_q(A^*_p) \leq 0 \) and is upper semicontinuous. Now \( \{ q : t_q(L) - t_q(A^*_p) = 0 \} = \{ q : t_q(L) - t_q(A^*_p) \geq 0 \} \), so this set is closed.)

Since \( W = \bigcup \{ Z_p : p \in W \} \) and a clopen set in \( N^* \) is not the union of \( c \) nowhere dense closed sets, at least one of the \( Z_p \) has nonvoid interior. (The number of distinct \( Z_p \) sets is at most \( c \) because this is true of the corresponding \( A_p \) sets.) Let \( A \subseteq N \) be such that \( A^* \subseteq Z_p \), and let \( E \) be the element \( A_p \) corresponding to \( Z_p \).

3.7. Proof that (a) implies (b). Let \( A = \{ n(k) \} \) and \( E \) be as in 3.6. If the matrix operator \( R \) is defined as in the statement of the main theorem, then it is clear that for all \( B \in F' \), \( R\lim 1_B = 1 \), and it follows easily that the support of \( R \) is contained in \( L \). This completes the proof.

References


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