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ON ABSOLUTE WEIGHTED MEAN SUMMABILITY METHODS

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ABSTRACT. In this paper we have proved the converse of the Bor-Thorpe theorem [2] on absolute weighted mean summability.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums s_n, and let $A = (a_{nv})$ be a normal matrix, that is, lower-semi matrix with nonzero entries. By $(A_n(s))$ we denote the A-transform of the sequence $s = (s_n)$, i.e.,

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v .$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} |A_n(s) - A_{n-1}(s)|^k < \infty ,$$

[4]. In the special case when A is a Riesz matrix (resp. $k = 1$), $|A|_k$ summability is the same as $|\overline{N}, p_n|_k$ (resp. $|\overline{N}, p_n|$) summability (see [1]). By a Riesz matrix we mean one such that

$$a_{nv} = \frac{p_v}{P_n} \quad \text{for } 0 \leq v \leq n ,$$
$$a_{nv} = 0 \quad \text{for } n > v ,$$

where (p_n) is a sequence of positive real constants for which $P_n = p_0 + p_1 + \cdots + p_n \neq 0$, $P_n \to \infty$ as $n \to \infty$, $P_{-1} = p_{-1} = 0$.

If P and Q are methods of summability, Q is said to include P if every series summable by the method P is also summable by the method Q. P and Q are said to be equivalent (written "$P \Leftrightarrow Q$") if each method includes the other.

Dealing with the absolute weighted mean summability of infinite series, Bor and Thorpe [2] established the following generalization of the Theorem due to Bor [1].

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Theorem A. *Suppose* (p_n) *and* (q_n) *are positive sequences such that*

$$\text{(1.1)} \qquad \text{(i) } p_n Q_n = O(P_n q_n), \quad \text{(ii) } P_n q_n = O(p_n Q_n).$$

Then $|\overline{N}, p_n|_k \Leftrightarrow |\overline{N}, q_n|_k$, $k \geq 1$.

2. The main result

The object of this paper is to prove the converse of Theorem A as follows.

Theorem B. *Suppose* (p_n) *and* (q_n) *are positive sequences. If* $|\overline{N}, p_n|_k \Leftrightarrow |\overline{N}, q_n|_k$, $k \geq 1$, *then* (1.1) *holds.*

For the proof of the theorem, we require the following

Lemma. *Suppose that* $k > 0$ *and* $p_n > 0$, $P_n \to \infty$ *as* $n \to \infty$. *Then there exists two (strictly) positive constants* M *and* N, *depending only on* k, *for which for all* $v \geq 1$

$$\text{(2.1)} \qquad \frac{M}{P_{v-1}^k} \leq \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq \frac{N}{P_{v-1}^k},$$

where M *and* N *are independent of* (p_n).

Proof. Consider the function $f\colon [0, 1) \to \mathbb{R}$ given by $f(x) = (1-x)^{-1} \times (1-x^{1/k})$. It is obvious that this is continuous and strictly positive. Also, $f(x) \to 1/k$ as $x \to 1-0$. Thus there must be two positive constants M and N, depending only on k, for which

$$\text{(2.2)} \qquad M \leq f(x) \leq N \quad \text{for all } 0 \leq x < 1.$$

Now put $x = (P_{n-1}/P_n)^k = (1 - p_n/P_n)^k$ in (2.2). Then we have, for $n \geq 1$,

$$\text{(2.3)} \qquad M\left(\frac{1}{P_{n-1}^k} - \frac{1}{P_n^k}\right) \leq \frac{p_n}{P_n P_{n-1}^k} \leq N\left(\frac{1}{P_{n-1}^k} - \frac{1}{P_n^k}\right).$$

Therefore (2.1) follows from (2.3), completing the proof of the lemma.

3. Proof of Theorem B

Let $(u_n(a))$ and $(t_n(a))$ be the sequences of $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ means of the series $a = \sum a_n$, respectively. Then by the definition, we have

$$\text{(3.1)} \qquad u_n(a) = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v,$$

$$U_n(a) = u_n(a) - u_{n-1}(a) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v,$$

$$P_{-1} = 0, \qquad U_0(a) = a_0,$$

and

$$t_n(a) = \frac{1}{Q_n}\sum_{v=0}^{n} q_v s_v, \tag{3.2}$$

$$T_n(a) = t_n(a) - t_{n-1}(a) = \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1} a_v,$$

$$Q_{-1} = 0, \qquad T_0(a) = a_0.$$

For $k \geq 1$, define $|\overline{N}_p|_k = \{(a_i):\ \sum a_i$ is summable $|\overline{N}, p_n|_k\}$ and $|\overline{N}_q|_k = \{(a_i):\ \sum a_i$ is summable $|\overline{N}, q_n|_k\}$. Then it is routine to verify that these are BK-spaces if normed by

$$\|a\|_1 = \left\{\sum_{n=0}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |U_n(a)|^k\right\}^{1/k} \tag{3.3}$$

and

$$\|a\|_2 = \left\{\sum_{n=0}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1} |T_n(a)|^k\right\}^{1/k}, \tag{3.4}$$

respectively. Since $|\overline{N}, p_n|_k \Leftrightarrow |\overline{N}, q_n|_k$ by the hypothesis of the theorem, $\|a\|_1 < \infty \Leftrightarrow \|a\|_2 < \infty$, i.e., $|\overline{N}_p|_k \equiv |\overline{N}_q|_k$. Now consider the inclusion map $r\colon |\overline{N}_p|_k \to |\overline{N}_q|_k$ defined by $r(x) = x$ and its inverse. These are continuous, which is immediate as $|\overline{N}_p|_k$ and $|\overline{N}_q|_k$, $k \geq 1$, are BK-spaces. Thus there exists constants H and K such that

$$H\|a\|_1 \leq \|a\|_2 \leq K\|a\|_1 \tag{3.5}$$

for all $a \in |\overline{N}_p|_k \equiv |\overline{N}_q|_k$. By applying (3.1) and (3.2) to $a = e_v - e_{v+1}$ (e_v is the vth coordinate vector), respectively, we have

$$U_n(a) = \begin{cases} 0 & \text{if } n < v, \\ p_v/P_v & \text{if } n = v, \\ -p_v p_n/P_n P_{n-1} & \text{if } n > v; \end{cases}$$

$$T_n(a) = \begin{cases} 0 & \text{if } n < v, \\ q_v/Q_v & \text{if } n = v, \\ -q_v q_n/Q_n Q_{n-1} & \text{if } n > v, \end{cases}$$

so that (3.3) and (3.4) gives

$$\|a\|_1 = \left\{\frac{p_v}{P_v} + p_v^k \sum_{n=v+1}^{\infty}\frac{p_n}{P_n P_{n-1}^k}\right\}^{1/k},$$

$$\|a\|_2 = \left\{\frac{q_v}{Q_v} + q_v^k \sum_{n=v+1}^{\infty}\frac{q_n}{Q_n Q_{n-1}^k}\right\}^{1/k}.$$

Hence it follows from (3.5) that

$$H^k\left\{\frac{p_v}{P_v}+p_v^k\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\right\}\le\frac{q_v}{Q_v}+q_v^k\sum_{n=v+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}^k}$$
$$\le M^k\left\{\frac{p_v}{P_v}+p_v^k\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\right\},$$

which holds if and only if (1.1) is satisfied by the lemma. This completes the proof.

Our final result follows from Theorems A and B.

Theorem C. *Suppose* (p_n) *and* (q_n) *are positive sequences with* $P_n \to \infty$ *and* $Q_n \to \infty$ *as* $n \to \infty$. *Then the necessary and sufficient condition for* $|\overline{N}, p_n|_k \Leftrightarrow |\overline{N}, q_n|_k$, $k \ge 1$, *is that* (1.1) *hold.*

We remark that if we take $q_n = 1$ for all n, then $Q_n = n+1$. In this case $|\overline{N}, q_n|_k$ summability is the same as $|C, 1|_k$ summability [3]. Therefore the following corollary can be derived from Theorem C.

Corollary. *Necessary and sufficient conditions for* $|C, 1|_k \Leftrightarrow |\overline{N}, p_n|_k$ *are that*

$$\text{(i) } np_n = O(P_n)$$
$$\text{(ii) } P_n = O(np_n).$$

The sufficiency and necessity of this result are proven in [1] and [5], respectively.

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