*-REPRESENTATIONS OF THE TRACE-CLASS OF AN H*-ALGEBRA

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Abstract. The aim of this note is to characterize the cyclic and the irreducible
*-representations of the trace-class of a proper H*-algebra.

Throughout this paper $A$ denotes a proper H*-algebra (i.e., $A$ is a Banach
*-algebra whose norm is a Hilbert space norm such that $\langle x, yz^* \rangle = \langle xz, y \rangle =
\langle z, x^*y \rangle$ for every $x, y, z \in A$). A projection in $A$ is a nonzero element $e$
of $A$ such that $e^2 = e = e^*$; $e$ is called primitive if it cannot be represented
as a sum of two mutually orthogonal primitive projections in $A$. A maximal
family of mutually orthogonal primitive projections is called projection base.
An element $a \in A$ is said to be positive ($a \geq 0$) if $\langle ax, x \rangle \geq 0$ for every
$x \in A$. For each $a \in A$ there exists a unique positive element $\|a\|$ of $A$ such
that $\|a\|^2 = a^*a$.

By the trace-class of $A$ we mean the set $\tau(A) = \{xy : x, y \in A\}$ that is
dense in $A$. If $a \in A$, then the following assertions are equivalent:

(i) $a \in \tau(A)$.
(ii) $\|a\| \in \tau(A)$.
(iii) There exists a positive element $b$ of $A$ such that $b^2 = \|a\|$.
(iv) $\sum a\langle a|e_\alpha, e_\alpha \rangle < \infty$ for some projection base $\{e_\alpha\}$ in $A$.

There is a positive linear functional $\tau$ (called trace) on $\tau(A)$ such that
$\tau(xy^*) = \tau y^*x = \langle x, y \rangle$ and $\tau a = \tau a^*$ for every $x, y \in A$ and $a \in \tau(A)$.
One can define a Banach algebra norm $\tau$ on $\tau(A)$ by the formula $\tau(a) = \|a\|$
($a \in \tau(A)$). Denote by $R(A)$ the set of right centralizers on $A$, i.e., let

$R(A) = \{S \in B(A) : S(xy) = (Sx)y \ (\forall x, y \in A)\}$,

where $B(A)$ denotes the set of bounded linear operators on $A$. It is trivial that
$L_x$, the operator of the left multiplication by $x$, is in $R(A)$ for every $x \in A$.
$R(A)$ is isomorphic and isometric to $\tau(A)^*$.

As for the detailed discussion of proper H*-algebras and their trace-classes
as well as the proofs of the above statements we refer to [1, 5, 6].

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A positive linear functional \( f \) on a Banach *-algebra \( B \) is called representable if there is a Hilbert space \( H \) and a *-representation \( x \mapsto T_x \) of \( B \) on \( H \) with cyclic vector \( b \in H \) such that \( f(x) = \langle T_x b, b \rangle \) (\( x \in B \)). In [2, Theorem 37.11] it was stated that a positive linear functional \( f : B \to \mathbb{C} \) is representable if and only if there exists a positive constant \( c \in \mathbb{R} \) for which

\[
|f(x)|^2 \leq c f(x^* x) \quad (x \in B).
\]

Unfortunately, the proof presented there is incomplete since it uses the hermicity of the functional. For a correct proof see [4].

We begin with the following two basic lemmas.

**Lemma 1.** Let \( S \in R(A) \). Then the following assertions are equivalent:

(i) \( \sum_a |S| e_a, e_a < \infty \) for some projection base \( \{e_a\} \) in \( A \).

(ii) There exists a unique \( a \in \tau(A) \) such that \( S = L_a \).

**Proof.** Suppose that (i) holds. From the inequality \( S^* S \leq \|S\|S \) we have \( \sum_a \|S e_a\|^2 < \infty \). Since \( S \) is a right centralizer, one can easily verify that \( \{S e_a\} \) is a mutually orthogonal vector system. Let \( a = \sum a S e_a \). Then

\[
L_a x = ax = (\sum a S e_a) x = S(\sum a e_a x) = S x \quad (x \in A),
\]

where we have used the fact that \( x = \sum a e_a x \) for every \( x \in A \). Now \( L_{|a|} = |L_a| = |S| \) implies that \( a \in \tau(A) \). The uniqueness of \( a \) is obvious.

The other implication is easy to prove.

**Lemma 2.** Let \( a \in \tau(A) \) be positive. Then

\[
\tau(a) = \inf \{c \in \mathbb{R} : c \geq 0, \quad |\text{tr} ax|^2 \leq c \text{tr} ax^* x \quad (x \in \tau(A))\}.
\]

**Proof.** Consider the semi-inner product \( B \) on \( \tau(A) \) defined by

\[
B(x, y) = (ax, y) \quad (x, y \in \tau(A)).
\]

The Cauchy-Schwarz inequality implies that

\[
|\langle ae, e \rangle|^2 = |\langle ax, e \rangle|^2 \leq \langle ae, e \rangle \langle ax, x \rangle \quad (x \in \tau(A)),
\]

where \( e \) is an arbitrary projection in \( A \). Now it follows that

\[
|\text{tr} ax^*|^2 = |\text{tr}(ax^*)|^2 = |\text{tr} ax|^2 \leq \tau(a) \text{tr} x^*ax = \tau(a) \text{tr} ax^*x \quad (x \in \tau(A)).
\]

If \( c \in \mathbb{R}, \ c \geq 0 \) such that \( |\text{tr} ax|^2 \leq c \text{tr} ax^* x \quad (x \in \tau(A)) \), then for every projection \( e \) in \( A \) we have \( \langle ae, e \rangle = \text{tr} ae \leq c \), which implies that \( \tau(a) \leq c \).

Our first theorem characterizes the representable positive linear functionals on \( \tau(A) \).

**Theorem 1.** Let \( f : \tau(A) \to \mathbb{C} \) be a positive linear functional. Then the following assertions are equivalent:

(i) \( f \) is representable.

(ii) There exists a unique positive element \( a \) of \( \tau(A) \) such that \( f(x) = \text{tr} ax \) for every \( x \in \tau(A) \).

(iii) There exists a unique positive element \( b \) of \( A \) such that \( f(x) = \langle L_x b, b \rangle \) for every \( x \in \tau(A) \).
Proof. Let \( f \) be representable. Then there is a positive constant \( c \in \mathbb{R} \) such that \( |f(x)|^2 \leq c f(x^*x) \) \( (x \in \tau(A)) \). Since \( f \in \tau(A)^* \), by [5, Theorem 2], there is a positive operator \( S \in R(A) \) for which \( f(x) = \text{tr} Sx \) \( (x \in \tau(A)) \). If \( e \in A \) is a projection, then we have \( |\text{tr} Se|^2 \leq c \text{tr} Se \), i.e., \( \text{tr} Se \leq c \). Since it holds for every projection in \( A \), we can conclude that \( \sum_n \langle Se_n, e_n \rangle \leq c \) for every projection base \( \{e_n\} \) in \( A \). By Lemma 1 there is a positive element \( a \) in \( \tau(A) \) such that \( S = L_a \). The uniqueness of \( a \) follows from the density of \( \tau(A) \) in \( A \).

To the implication \( (ii) \Rightarrow (iii) \), let \( b \in A \) be the positive square root of \( a \). The remainder part of the statement is easy to check.

As a consequence of the above theorem and [3, Lemma (4.5.8)] we have the following

**Theorem 2.** Let \( b \in A \). If \( H_b \) denotes the closure of the subspace \( \tau(A)b \) in \( A \) then \( x \mapsto L_x \mid H_b \) is a \(*\)-representation of \( \tau(A) \) with cyclic vector \( b \). Moreover, every cyclic \(*\)-representation of \( \tau(A) \) is unitarily equivalent to a representation of this kind.

**Proof.** The only thing that has to be proved is \( b \in H_b \) for every \( b \in A \). But it follows from the fact that the projections in \( A \) belong to \( \tau(A) \).

In what follows, let

\[
P = \{ f \in \tau(A)^* : f \text{ is positive and } |f(x)|^2 \leq f(x^*x) \text{ (}x \in \tau(A)\}) \}.
\]

By Theorem 1, for every representable positive linear functional \( f \) on \( \tau(A) \) there exists a unique positive member \( a \) of \( \tau(A) \) such that \( f(x) = \text{tr} ax \) \( (x \in \tau(A)) \). Now, by Lemma 2, \( f \in P \) if and only if \( f(a) \leq 1 \). If \( f \) is not identically zero, then by [3, Corollary (4.6.5)], one can easily verify that \( f \) is an extremal point of \( P \) if and only if the conditions \( a \in \tau(A) \), \( a \geq 0 \), \( \tau(a) \leq 1 \), and \( \lambda a - a \geq 0 \) for some \( 0 < \lambda \leq 1 \) imply that there is a \( 0 \leq \mu \in \mathbb{R} \) such that \( \mu a = a \).

**Theorem 3.** Let \( 0 \neq f \in P \) and \( a \) be the unique element of \( \tau(A) \) corresponding \( f \) as above. Then \( f \) is an extremal point of \( P \) if and only if there exists a primitive projection \( e \) in \( A \) for which \( a = e/\|e\|^2 \).

**Proof.** Necessity. Suppose that \( f \) is an extremal point of \( P \). It is easy to see that \( \tau(a) = 1 \). Let \( a = \sum_n \lambda_n e_n \) be the spectral representation of \( a \) where \( 0 < \lambda_n \in \mathbb{R} \) and \( \{e_n\} \) is a sequence of mutually orthogonal primitive projections (see [6, Corollary 1]). Let \( \tilde{a} = e_1/\|e_1\|^2 \). Then \( \tilde{a} \in \tau(A) \), \( \tilde{a} \geq 0 \), and \( \tau(\tilde{a}) = 1 \). Moreover, for \( \lambda = 1/\|e_1\|^2 \) we have \( \lambda a - a \geq 0 \). Consequently, there exists an \( 0 \leq \mu \in \mathbb{R} \) such that \( \mu a = e_1/\|e_1\|^2 \). Taking traces we arrive at

\[
\mu = \mu \text{tr} a = (1/\|e_1\|^2) \text{tr} e_1 = 1.
\]

Sufficiency. Let \( a = e/\|e\|^2 \) where \( e \) is a primitive projection in \( A \). Suppose that \( a \in \tau(A) \), \( 0 \neq a \geq 0 \) such that \( \tau(a) \leq 1 \) and \( \lambda a - a \geq 0 \) for some \( 0 < \lambda \in \mathbb{R} \). Let \( \tilde{a} = \sum_n \lambda_n e_n \) be the spectral representation of \( a \). Then, for every fixed \( n \), we have \( \lambda e_n/\|e\|^2 \geq \lambda_n e_n \). If we extend the singleton \( \{e\} \) to a projection base, then the first structure theorem of proper \( H^* \)-algebras (c.f. [1, Theorem 4.1]) implies that \( e_n A \subset eA \). Since \( eA \) is a minimal closed right ideal thus \( e_n A = eA \). It is known that the projection of \( x \in A \) on the closed
right ideal $eA$, where $e$ is an arbitrary projection in $A$, is $ex$. Consequently, we have $e_n = ee_n = e_n e = e$, which implies that there is a $0 < \mu \in \mathbb{R}$ for which $\mu a = a$. This completes the proof.

Using the notation of Theorem 2, by [3, Theorem (4.6.4)], it is easy to prove our final result, which follows.

**Theorem 4.** Let $e$ be a primitive projection in $A$. Then $x \mapsto L_x \upharpoonright H_e$ is a nonzero irreducible *-representation of $\tau(A)$. Moreover, every irreducible *-representation of $\tau(A)$ is unitarily equivalent to a representation of this kind.

**References**


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