THE NUMERICAL RADIUS OF A NILPOTENT OPERATOR ON A HILBERT SPACE

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Abstract. Let $T$ be a bounded linear operator of norm 1 on a Hilbert space $H$ such that $T^n = 0$ for some $n \geq 2$. Then its numerical radius satisfies $w(T) \leq \cos \frac{n}{n+1}$ and this bound is sharp. Moreover, if there exists a unit vector $\xi \in H$ such that $|\langle T\xi | \xi \rangle| = \cos \frac{n}{n+1}$, then $T$ has a reducing subspace of dimension $n$ on which $T$ is the usual $n$-shift. The proofs show that these facts are related to the following result of Fejer: if a trigonometric polynomial $f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}$ is positive, one has $|f_1| \leq f_0 \cos \frac{n}{n+1}$; moreover, there is essentially one polynomial for which equality holds.

Let $H$ be a complex Hilbert space, possibly finite dimensional, and let $T$ be a (bounded linear) operator on $H$. The numerical radius of $T$ is

$$w(T) = \text{Sup}\{|\langle T\xi | \xi \rangle| : \xi \in H_1\}$$

where $H_1$ denotes the unit sphere in $H$. It is well known that

$$r(T) \leq w(T) \leq \|T\|$$

where $r(T)$ and $\|T\|$ denote respectively the spectral radius and the norm of $T$. In particular $w(T) = \|T\|$ whenever $T$ is normal. Also, it follows from polarization that

$$w(T) \geq \frac{1}{2} \|T\|.$$

For all this and much more, see [Ha1, §18] and [Ha2, Chapter 17]. The subject of this paper is the comparison of $w(T)$ with $\|T\|$ when $T$ is nilpotent, namely when $T^n = 0$ for some integer $n \geq 2$. We show in this case that $w(T) \leq \|T\| \cos \frac{n}{n+1}$, and equality holds for the $n$-shift on $\mathbb{C}^n$ (cf. Theorem 1). We also show how this is related to a result of Fejer about trigonometric polynomials of the form

$$f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}$$

with $f_k \in \mathbb{C}$. Such a polynomial is positive if $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$. The computation of the numerical radius for the $n$-shift implies that $|f_1| \leq f_0 \cos \frac{n}{n+1}$ (cf. Theorem 2).

Received by the editors September 7, 1990.

1991 Mathematics Subject Classification. Primary 47A12; Secondary 15A45, 42A05.

Key words and phrases. Numerical radius, nilpotent operator, positive trigonometric polynomial.

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0002-9939/92 $1.00 + $.25 per page
Theorem 1. Let $T$ be an operator on $H$ such that $T^n = 0$ for some $n \geq 2$.

1. One has

$$w(T) \leq \|T\| \cos \frac{\pi}{n+1}$$

and equality holds when $T$ is the $n$-dimensional shift on the standard hermitian space $\mathbb{C}^n$.

2. Suppose moreover that $\|T\| = 1$. Suppose that there exists a unit vector $\xi \in H_1$ with $|\langle T\xi|\xi\rangle| = \cos \frac{\pi}{n+1}$. (If $\dim H < \infty$, this holds if and only if $w(T) = \cos \frac{\pi}{n+1}$.) Let $V_\xi$ be the linear span of $\{\xi, T\xi, ..., T^{n-1}\xi\}$. Then $V_\xi$ is an $n$-dimensional subspace of $H$ which is reducing for $T$, and the restriction of $T$ to $V_\xi$ is unitarily equivalent to the $n$-dimensional shift on $\mathbb{C}^n$.

The proof uses inequalities on positive matrices (see Proposition 3), namely on matrices $\beta = (\beta_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbb{C})$ such that $\beta_{ij} \geq 0$ for all $i, j \in \mathbb{C}^n$; for such a matrix, we write $\beta \geq 0$.

From now on, $n$ is an integer, $n \geq 2$. For the proof of Theorem 1 we will need the following computation. (This is probably standard; see in particular [DH].)

Proposition 1. Let $S$ be the $n$-dimensional shift on $\mathbb{C}^n$, given by the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

and set $\xi_0 = \left(\frac{2}{n+1}\right)^{1/2} \sin \frac{k\pi}{n+1} \right)_{1 \leq k \leq n} \in \mathbb{C}^n$. Then

1. $\|S\| = 1$,
2. $w(S) = \cos \frac{\pi}{n+1}$,
3. for $\xi \in \mathbb{C}^n$ with $\|\xi\| = 1$, one has $\langle S\xi|\xi\rangle = \cos \frac{\pi}{n+1}$ if and only if $\xi = e^{i\phi}\xi_0$ for some $\phi \in \mathbb{R}$.

Proof. The first claim is obvious.

For any operator $T$ on any Hilbert space $H$, one has

$$w(T) = \sup\{|\Re(e^{i\theta}\langle T\xi|\xi\rangle) : \theta \in \mathbb{R} \text{ and } \xi \in H_1\}$$

$$= \frac{1}{2} \sup\{|(e^{i\theta}T + e^{-i\theta}T^*)\xi|\xi\rangle : \theta \in \mathbb{R} \text{ and } \xi \in H_1\}$$

$$= \frac{1}{2} \sup\{|e^{i\theta}T + e^{-i\theta}T^*| : \theta \in \mathbb{R}\}. $$

Set now

$$A = S + S^* = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. $$
If \( D(\theta) \) denotes the unitary diagonal matrix with entries \( e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta} \), one has
\[
D(\theta)^*(e^{i\theta} S + e^{-i\theta} S^*)D(\theta) = A
\]
for all \( \theta \in \mathbb{R} \). Consequently \( w(S) = \frac{1}{2} \|A\| \).

For \( k = 1, \ldots, n \), let \( \xi_k \in \mathbb{C}^n \) be the vector with coordinates
\[
\sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, \ldots, \sin \frac{nk\pi}{n+1}.
\]
One has
\[
A\xi_k = 2 \cos \left( \frac{k\pi}{n+1} \right) \xi_k, \quad k = 1, \ldots, n.
\]
(This computation appears already in \([\text{Lag}, \text{Nos. 19-20}]\).) Consequently
\[
w(S) = \frac{1}{2} \|A\| = \frac{1}{2} \sup_{1 \leq k \leq n} \left| 2 \cos \left( \frac{k\pi}{n+1} \right) \right| = \cos \left( \frac{\pi}{n+1} \right),
\]
so that Claim (2) holds. Observe that
\[
\|\xi_i\|^2 = \sum_{k=1}^{n} \sin^2 \left( \frac{k\pi}{n+1} \right) = \frac{n+1}{2},
\]
so that
\[
\xi_0 = \xi_1/\|\xi_1\|^{-1}.
\]
Choose now \( \xi \in \mathbb{C}^n \) with \( \|\xi\| = 1 \). As \( (\xi_k \|\xi_k\|^{-1})_{1 \leq k \leq n} \) is an orthonormal basis of \( \mathbb{C}^n \), there are complex numbers \( c_1, \ldots, c_n \) with \( \sum |c_k|^2 = 1 \) and
\[
\xi = \sum c_k \xi_k \|\xi_k\|^{-1}.
\]
Assume that \( \langle S\xi|\xi \rangle = \cos \frac{\pi}{n+1} \). Then
\[
\cos \frac{\pi}{n+1} = \frac{1}{2} \langle A\xi|\xi \rangle = \sum_{k=1}^{n} |c_k|^2 \cos \frac{k\pi}{n+1}.
\]
This implies that \( |c_1| = 1 \) and \( c_2 = \cdots = c_n = 0 \), so that \( \xi = c_1 \xi_0 \). This shows (3). \( \square \)

**Notation.** If \( \beta = (\beta_{i,j})_{1 \leq i, j \leq n} \in M_n(\mathbb{C}) \) is any given matrix, and if \( k, l \) are positive integers with \( \max\{k, l\} > n+1 \), we set \( \beta_{k,l} = 0 \).

**Lemma 1.** Consider an operator \( T \) on \( H \) such that \( T^n = 0 \) and \( \|T\| \leq 1 \), and a unit vector \( \xi \in H_1 \). Define
\[
\beta_{i,j} = (T^{i-1} \xi|T^{j-1} \xi), \quad i, j \geq 1.
\]
Then the matrix \( (\beta_{i,j} - \beta_{i+1,j+1})_{1 \leq i, j \leq n} \) is positive.

**Proof.** Consider complex numbers \( c_1, \ldots, c_n \) and set
\[
\eta = \sum_{k=1}^{n} c_k T^{k-1} \xi.
\]
One has \( 1 - TT^* \geq 0 \), because \( \|T\| \leq 1 \), and thus
\[
\langle (1 - T^*T)\eta|\eta \rangle = \sum_{i,j=1}^{n} c_i c_j (\beta_{i,j} - \beta_{i+1,j+1}) \geq 0.
\]
As this holds for any choice of \( c_1, \ldots, c_n \), the lemma follows. \( \square \)
Definition. With $S$ and $\xi_0$ as in Proposition 1, we define a matrix $\alpha = (\alpha_{i,j})_{1 \leq i, j \leq n}$ by

$$\alpha_{i,j} = \langle S^{-1} \xi_0 | S^{-1} \xi_0 \rangle.$$ 

Observe that $\alpha_{1,1} = \|\xi_0\|^2 = 1$, and that

$$\alpha_{1,2} = \alpha_{2,1} = \cos \frac{\pi}{n+1}$$

by Proposition 1.3. Observe also that $(\alpha_{i,j} - \alpha_{i+1,j+1})_{1 \leq i, j \leq n} \geq 0$ by Lemma 1.

Proposition 2. Let $\beta = (\beta_{i,j})_{1 \leq i, j \leq n} \in M_n(\mathbb{C})$ be a matrix such that $\beta_{1,1} = 1$. Assume that the matrix $\gamma \in M_n(\mathbb{C})$ defined by

$$\gamma_{i,j} = \beta_{i,j} - \beta_{i+1,j+1}$$

is positive. Then

1. $|\beta_{1,2}| \leq \alpha_{1,2} = \cos \frac{\pi}{(n+1)}$.

2. If moreover $\beta_{1,2} = \alpha_{1,2}$, then $\beta = \alpha$.

Proof. From the definition of $\gamma$, one has

$$\beta_{i,j} = \gamma_{i,j} + \gamma_{i+1,j+1} + \cdots + \gamma_{i+k,j+k}$$

with $k = \min\{n-i, n-j\}$. In particular

(I) \hspace{1cm} \text{trace}(\gamma) = \beta_{1,1} = 1

and $\beta_{1,2} = \gamma_{1,2} + \gamma_{2,3} + \cdots + \gamma_{n-1,n}$. As $\gamma$ is positive, $\gamma_{i,i}$ is real nonnegative for $i = 1, \ldots, n$ and $|\gamma_{i,j}|^2 \leq \gamma_{i,i} \gamma_{j,j}$ for $i, j = 1, \ldots, n$. Consequently

(II) \hspace{1cm} |\beta_{1,2}| \leq (\gamma_{1,1})^{1/2} (\gamma_{2,2})^{1/2} + \cdots + (\gamma_{n-1,n-1})^{1/2} (\gamma_{n,n})^{1/2}.

Let $\eta \in \mathbb{C}^n$ be the vector with coordinates $(\gamma_{1,1})^{1/2}, \ldots, (\gamma_{n,n})^{1/2}$. Then $\|\eta\|^2 = 1$ by (I) and $|\beta_{1,2}| = \langle S \eta | \eta \rangle$ by (II), so that Claim (1) follows by Proposition 1.

Recall that a state on $M_n(\mathbb{C})$ is a linear form $\omega$ on $M_n(\mathbb{C})$ such that $\omega(1) = 1$ and $\omega(T) \geq 0$ whenever $T \geq 0$. Each positive matrix $\delta \in M_n(\mathbb{C})$ with $\text{trace}(\delta) = 1$ defines a state $\omega_{\delta}$ by $\omega_{\delta}(T) = \text{trace}(T \delta)$. In particular, the matrix $\gamma$ defines a state $\omega_\gamma$ and one has

$$\omega_\gamma(S) = \sum_{i,j = 1}^n S_{i,j} \gamma_{i,j} = \sum_{j=1}^n \gamma_{j,j+1} = \beta_{1,2}.$$ 

Given vectors $\xi_1, \xi_2 \in \mathbb{C}^n$, we denote by $\xi_1 \otimes \xi_2$ the matrix of the linear endomorphism $\xi \mapsto \langle \xi | \xi_2 \rangle \xi_1$ of $\mathbb{C}^n$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of $\gamma$ and choose a basis $(\eta_\nu)_{1 \leq \nu \leq n}$ of corresponding eigenvectors, so that

$$\gamma = \sum_{\nu=1}^n \lambda_\nu \eta_\nu \otimes \overline{\eta_\nu}$$

and

$$S \gamma = \sum_{\nu=1}^n \lambda_\nu S \eta_\nu \otimes \overline{\eta_\nu}.$$
The hypothesis for Claim (2) reads now
\[ \omega_\gamma(S) = \text{trace}(S\gamma) = \sum_{\nu=1}^{n} \lambda_\nu \langle S\eta_\nu | \eta_\nu \rangle = \cos \frac{\pi}{n + 1}. \]

As \( \sum_{\nu=1}^{n} \lambda_\nu = \text{trace}(\gamma) = 1 \), it follows from Proposition 1 that
\[ \lambda_1 = 1, \quad \lambda_2 = \ldots = \lambda_n = 0, \]
and that \( \eta_1 = e^{i\phi} \xi_0 \) for some \( \phi \in \mathbb{R} \). This can be expressed as
\[ \gamma = \xi_0 \otimes \overline{\xi_0}. \]

Denote by \( \xi_0^1, \ldots, \xi_0^n \) the coordinates of \( \xi_0 \). One has
\[ \beta_{i,j} = \sum_{k=0}^{n} \xi_0^{i+k} \xi_0^{j+k}, \]
where \( k = \min\{n - i, n - j\} = n - \max\{i, j\} \). But \( \xi_0^p = \xi_0^{n+1-p} \) for all \( p \in \{1, \ldots, n\} \) and
\[ \beta_{i,j} = \sum_{k=0}^{n} (S^j \xi_0^k)(S^j \xi_0^k) = \langle (S^j \xi_0^k)(S^j \xi_0^k) = \alpha_{i,j} \]
for all \( i, j \in \{1, \ldots, n\} \).

**Lemma 2.** Let \( T \) be an operator on \( H \) such that \( \|T\| = 1 \) and \( T^n = 0 \), and assume that there exists a unit vector \( \xi \in H_1 \) such that
\[ \langle T^k \xi | \xi \rangle = \cos \frac{\pi}{n + 1}. \]
Define \( V_\xi \) to be the subspace of \( H \) spanned by \( \xi, T \xi, \ldots, T^{n-1} \xi \). Then

1. \( (T^*)^{k-1} \xi \in V_\xi \) for \( k = 1, \ldots, n \).
2. \( V_\xi \) is a reducing subspace for \( T \).

**Proof.** Lemma 1 and Proposition 2 imply that
\[ \langle T^i \xi | T^j \xi \rangle = \langle (S^j \xi_0)(S^j \xi_0) \]
for all \( i, j \in \{1, \ldots, n\} \), and in particular \( V_\xi \) is isometrically isomorphic to the span of \( \xi_0, S\xi_0, \ldots, S^{n-1} \xi_0 \), which is all of \( \mathbb{C}^n \), so it follows that \( V_\xi \) is of dimension \( n \).

Define \( \beta \in M_n(\mathbb{C}) \) by \( \beta_{i,j} = \langle (T^*)^{i-1} \xi | (T^*)^{j-1} \xi \rangle \). Lemma 1 and Proposition 2 imply also that
\[ \beta = \alpha \]
and in particular that \( \|(T^*)^{i-1} \xi\| = (\alpha_{i,i})^{1/2} \) for \( i = 1, \ldots, n \).

Let \( P \) be the orthogonal projection from \( H \) onto \( V_\xi \). Consider some \( k \in \{1, \ldots, n\} \), and let \( c_1, \ldots, c_n \) be the (uniquely defined) complex numbers such that
\[ P(T^*)^{k-1} \xi = c_1 \xi + c_2 T \xi + \cdots + c_n T^{n-1} \xi. \]
As \((T^\nu-\xi)\chi\leq\nu\leq n\) is a basis of \(V\) (though not an orthonormal one!) these \(c_\nu\)'s can be written as functions of the numbers
\[
\langle(T^*)^{k-1}\xi|T^{i-1}\xi\rangle \quad \text{and} \quad \langle T^{i-1}\xi|T^{j-1}\xi\rangle, \quad i, j = 1, \ldots, n.
\]
But (IV) shows that
\[
\langle(T^*)^{k-1}\xi|T^{i-1}\xi\rangle = \langle(T^*)^{k+i-2}\xi|\xi\rangle = \langle((S^*)^{k-1}\xi_0|S^{i-1}\xi_0\rangle
\]
for \(i = 1, \ldots, n\) and (III) reads
\[
\langle T^{i-1}\xi|T^{j-1}\xi\rangle = \langle S^{i-1}\xi_0|S^{j-1}\xi_0\rangle
\]
for \(i, j = 1, \ldots, n\). It follows that the coordinates of \((S^*)^{k-1}\xi_0\) with respect to the non-orthogonal basis \(\xi_0, S\xi_0, \ldots, S^{n-1}\xi_0\) of \(\mathbb{C}^n\) are also \(c_1, c_2, \ldots, c_n\), i.e.
\[
(S^*)^{k-1}\xi_0 = c_1\xi_0 + c_2S\xi_0 + \cdots + c_nS^{n-1}\xi_0,
\]
hence
\[
\|P(T^*)^{k-1}\xi\| = \|(S^*)^{k-1}\xi_0\| = (\alpha_{k,k})^{1/2} = \|(T^*)^{k-1}\xi\|.
\]
Consequently \((T^*)^{k-1}\xi \in \text{Im}(P) = V\) and this proves (1).

Define \(W_a\) to be the linear span of \(\xi, T^a\xi, \ldots, (T^*)^{a-1}\xi\). One checks as for \(V\) that \(W_a\) has dimension \(n\), and Claim (1) shows that \(W_a \subset V\). Hence \(W_a = V\). As \(V\) [respectively \(W_a\)] is obviously invariant by \(T\) [resp. \(T^*\)], it follows that \(V\) is invariant by \(T\) and \(T^*\). □

**Proof of Theorem 1.** Let \(T\) be an operator on \(H\) such that \(T^n = 0\). One may assume that \(\|T\| = 1\).

(1) Choose a vector \(\xi \in H_1\) and define \(\beta \in M_n(\mathbb{C})\) by
\[
\beta_{i,j} = \langle T^{i-1}\xi|T^{j-1}\xi\rangle.
\]
By Lemma 1 and Proposition 2.1, one has \(|\beta_{1,2}| \leq \cos \frac{\pi}{n+1}\). As this holds for all \(\xi \in H_1\), this shows that \(w(T) \leq \cos \frac{\pi}{n+1}\).

(2) By hypothesis, there exist \(\xi \in H_1\) and \(\theta \in \mathbb{R}\) such that \(\langle T^a\xi|\xi\rangle = e^{i\theta} \cos \frac{\pi}{n+1}\). Let \(V_a\) be the linear span of \(\xi, T\xi, \ldots, T^{n-1}\xi\).

Define \(T' = e^{-i\theta}T\). Then \(V_a\) is a reducing subspace for \(T'\) by Lemma 2 and the restriction of \(T'\) to \(V_a\) is unitarily equivalent to the \(n\)-dimensional shift \(S\) by Proposition 2. Hence \(V_a\) is reducing for \(T\) and the restriction of \(T\) to \(V_a\) is unitarily equivalent to \(e^{i\theta}S\). But \(D(\theta)e^{i\theta}SD(\theta)^* = S\) if \(D(\theta)\) is the unitary diagonal matrix defined in the proof of Proposition 1, so that the proof is complete. □

From Theorem 1, it is easy to deduce the following 1915 result of L. Fejer (see [PS, Problem VI.52]). We are grateful to J. Steinig who brought Fejer’s result to our attention.

**Theorem 2.** Consider a positive integer \(n \geq 2\) and a positive trigonometric polynomial
\[
f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}
\]
of degree at most \(n - 1\). Assume that \(f \neq 0\), so that in particular \(f_0 > 0\).
(1) One has $|f_1| = f_0 \cos \frac{\pi}{n+1}$, and the constant $\cos \frac{\pi}{n+1}$ is the best possible one.

(2) Suppose moreover that $|f_1| = f_0 \cos \frac{\pi}{n+1}$. Then

$$f(\theta) = f_0 |g(\theta + \psi)|^2 \text{ for all } \theta \in \mathbb{R}$$

where $g$ is defined by

$$g(\phi) = \left(\frac{2}{n+1}\right)^{1/2} \sum_{k=0}^{n-1} \sin \left(\frac{(k+1)\pi}{n+1}\right)e^{ik\phi}, \quad \phi \in \mathbb{R},$$

and where $\psi = \arg(f_1)$.

Proof. As $f \neq 0$ and $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$, one has

$$f_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta > 0,$$

and there is no loss of generality if we assume that $f_0 = 1$. A theorem of Fejer and Riesz [Rud, Theorem 8.4.5] shows that there exists a trigonometric polynomial $g(\theta) = \sum_{k=0}^{n-1} g_k e^{ik\theta}$ such that

$$f(\theta) = |g(\theta)|^2 = \sum_{k=-n+1}^{n-1} (\sum_{p,q \in \{0,1,\ldots,n-1\}} g_p \bar{g}_q)e^{ik\theta}$$

for all $\theta \in \mathbb{R}$, where the second summation is over $p, q \in \{0,1,\ldots,n-1\}$ with $p - q = k$. One has in particular

$$\sum_{p=0}^{n-1} |g_p|^2 = 1 \quad \text{and} \quad f_1 = \langle S \eta | \eta \rangle$$

where $\eta \in \mathbb{C}^n$ is the unit vector with coordinates $g_0, \ldots, g_{n-1}$. Thus $|f_1| \leq w(S) = \cos \frac{\pi}{n+1}$ by Proposition 1.2.

Suppose now that $|f_1| = \cos \frac{\pi}{n+1}$. Upon replacing $f$ by the polynomial $\theta \mapsto f(\theta - \beta)$ for some $\beta \in \mathbb{R}$, one may assume that $f_1 = \cos \frac{\pi}{n+1}$. It follows from Proposition 1.3 that $\eta = e^{i\phi} \xi_0$ for some $\phi \in \mathbb{R}$. There is no change in (V) if we replace $g$ by $e^{-i\phi}g$, so that we may assume that $\eta = \xi_0$. But then $g$ has precisely the form given in Theorem 2.2. \qed

Remarks. (a) Let $T$ be as in Theorem 1 and such that $\|T\| = 1$. For each $\zeta \in H_1$ and $\theta \in \mathbb{R}$, set

$$f_{\zeta,k} = \begin{cases} 1 & \text{if } k = 0, \\ \langle T^k \zeta | \zeta \rangle & \text{if } k > 0, \\ \langle \zeta | T^{|k|} \zeta \rangle & \text{if } k < 0, \end{cases}$$

and

$$f_{\zeta}(\theta) = \sum_{k=-n+1}^{n-1} f_{\zeta,k} e^{ik\theta}.$$
Set also
\[ X = 1 + \sum_{k \geq 1}((e^{i\theta} T)^k + (e^{-i\theta} T^*)^k) \]
\[ = (1 - e^{i\theta} T)^{-1} + (1 - e^{-i\theta} T^*)^{-1} - 1 \]
\[ = (1 - e^{-i\theta} T^*)^{-1} \{ 1 - e^{-i\theta} T^* + 1 - e^{i\theta} T \}
- (1 - e^{-i\theta} T^*)(1 - e^{i\theta} T) \} (1 - e^{i\theta} T)^{-1} \]
and
\[ \eta = (1 - e^{i\theta} T)^{-1} \zeta. \]

Then
\[ f_\zeta(\theta) = \langle X \zeta | \zeta \rangle = \langle (1 - T^* T) \eta | \eta \rangle \geq 0 \]
because \( \|T\| = 1 \). Using now Fejer's Theorem 2.1, we see that
\[ |\langle T \zeta | \zeta \rangle| \leq \cos \frac{\pi}{n+1} \]
for all \( \zeta \in H_1 \). This provides a new proof of the inequality in Theorem 1.1.

(b) Assume now that \( n \geq 3 \). For each \( t \in [0, 1] \), set
\[ S(t) = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & t & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & t & 0
\end{pmatrix} \in M_n(\mathbb{C}) \]
and let \( w(t) \) be the numerical radius of \( S(t) \). One has \( \|S(t)\| = 1 \) for all \( t \in [0, 1] \). One has also \( w(1) = \cos \frac{\pi}{n+1} \) and \( w(0) = \frac{1}{2} \). As \( w(t) \) is continuous in \( T \) (see [Ha2, Problem 175]), the set of possible numerical radii of nilpotent operators of norm 1 of order at most \( n \) is \( \left[ \frac{1}{2}, \cos \frac{\pi}{n+1} \right] \).

On the other hand, let \( T \) be an operator of norm 1 on a finite dimensional Hilbert space \( H \) (or more generally on a Hilbert space in which there exists a unit vector \( \xi \) such that \( \|T \xi\| = 1 \)), and assume that \( w(T) = \frac{1}{2} \). Then it is known that \( T \) has a reducing subspace of dimension 2 on which it is unitarily equivalent to the 2-shift. (See [WC]; we are grateful to A. Sinclair who made us aware of this.)

For \( n \geq 3 \), let \( I_n \) be the set of those positive real numbers \( t \) for which there exists an operator \( T \) acting on a finite dimensional Hilbert space with the following properties: \( t = w(T) \), \( \|T\| = 1 \), \( T^n = 0 \), \( T^{n-1} \neq 0 \), and \( T \) is irreducible (no nontrivial reducing subspace). The two facts just above show that \( I_n = \left[ \frac{1}{2}, \cos \frac{\pi}{n+1} \right] \).

(c) Let \( S(t) \) and \( w(t) \) be as above, now with \( 0 < t < 1 \). It is easy to check that \( w(t) < \cos \frac{\pi}{n+1} \). Let \( (t_\nu)_{\nu \geq 1} \) be a strictly increasing sequence of positive numbers converging to 1. Consider an operator on an infinite dimensional Hilbert space which is the orthogonal sum of the \( S(t_\nu) \)'s. Such an operator shows that the hypothesis that \( T \) attains its numerical radius cannot be omitted in Claim (2) of Theorem 1.
We have illustrated one more appearance of the ubiquitous sequence

\[
\left( \cos \frac{\pi}{n + 1} \right)_{n \geq 2}.
\]

Its recent popularity is due to the work of V. Jones on index for subfactors and all that [Jon], but it appears in many other domains, such as the elementary geometry of regular polygons [Cox, Formula 2.84], graph theory or Fuchsian groups [GHJ], to mention but a few.

The second author wishes to note that the original idea of this paper is due to the first author. The paper itself was written up while we were both enjoying the hospitality of the Mittag-Leffler Institute during the autumn 1988. We are grateful to A. Sinclair and J. Steinig for their useful comments.

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