FIXED POINTS FOR DISCONTINUOUS QUASIMONOTONE MAPS IN SEQUENCE SPACES

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Abstract. In [2] Hu gives a fixed point theorem for discontinuous quasimonotone increasing maps in \( X = \mathbb{R}^n \). We will answer the question in [2] as to whether this result can be extended to \( X = l^p \), \( 1 \leq p \leq \infty \).

Let the Banach spaces \( X = \mathbb{R}^n \), \( c_0 \), and \( l^p \), \( 1 \leq p \leq \infty \), be ordered by the cone \( K = \{ x = (x_i)_{i \in I} \in X : x_i \geq 0 \} \), where \( I = \{1, \ldots, n\} \) or \( I = \mathbb{N} \), respectively. Then \( (X, K) \) is order complete, i.e., if a subset \( A \) of \( X \) has an upper bound, then \( A \) has a least upper bound, which we denote by \( \sup A \).

For \( u, v \in X \), \( u \leq v \), we set \( [u, v] = \{ z \in X : u \leq z \leq v \} \). It is well known (theorem of Tarski; see [4]) that for each increasing map \( M : [u, v] \to [u, v] \) the points \( \bar{x} = \sup \{ x \in [u, v] : x \leq Mx \} \) and \( \underline{x} = \inf \{ x \in [u, v] : x \geq Mx \} \) are fixed points of \( M \).

For a function \( f = (f_i)_{i \in I} : X \to X \) we define for \( x \in X \), in analogy to the notation in [2],

\[
D_\pm f_i(x) = \lim_{t \to 0^\pm} \frac{1}{t} (f_i(x + te_i) - f_i(x)),
\]

where \( e_i \), \( i \in I \), are the elements of \( X \) with components \( e_{ij} = 1 \) for \( i = j \), \( e_{ij} = 0 \) for \( i \neq j \).

Theorem. Let \( u, v \in X \), \( u \leq v \), and \( f = (f_i)_{i \in I} : [u, v] \to X \) be a function with the following properties:

1. \( f_i(x) \leq f_i(y) \) for \( x, y \in [u, v] \) with \( x \leq y \) and \( x_i = y_i \), \( i \in I \);
2. \( \min \{ D_- f_i(x) \}, D_+ f_i(x) \} > -\infty \) for \( x \in [u, v] \), \( i \in I \);
3. \( u_i \leq f_i(x + (u_i - x_i)e_i) \), \( v_i \geq f_i(x + (v_i - x_i)e_i) \) for \( x \in [u, v] \), \( i \in I \).

Then \( f \) has a greatest fixed point \( \bar{x} \) and a smallest fixed point \( \underline{x} \), and

\[
\bar{x} = \sup \{ x \in [u, v] : x \leq f(x) \}, \quad \underline{x} = \inf \{ x \in [u, v] : x \geq f(x) \}.
\]

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In case $X = \mathbb{R}^n$ this result is exactly [2, Theorem 2].

Proof. For each $x \in X$, $i \in I$ define

$$g_i^x(t) = f_i(x + (t - x_i)e^i) \quad \text{for } t \in [u_i, v_i].$$

From (2) and (3) we get $\min\{D_-g_i^x(t), D_+g_i^x(t)\} > -\infty$ for $t \in [u_i, v_i]$ and $u_i \leq g_i^x(u_i)$, $v_i \geq g_i^x(v_i)$.

Hence by [2, Corollary 1] the function $g_i^x(t)$ has a greatest fixed point $M_i x$ and a smallest fixed point $m_i x$ and

$$M_i x = \sup\{t \in [u_i, v_i] : t \leq g_i^x(t)\},$$

$$m_i x = \inf\{t \in [u_i, v_i] : t \geq g_i^x(t)\}.$$ 

Since, by (1), $g_i^x(t) \leq g_j^x(t)$ for each pair $x$, $y \in [u, v]$ with $x \leq y$, we conclude that $M_i x \leq M_j y$, $m_i x \leq m_j y$ for $x$, $y \in [u, v]$ with $x \leq y$. Therefore $M x = (M_i x)_{i \in I}$ defines an increasing map $M : [u, v] \to [u, v]$.

Let $\bar{x} = \sup\{x \in [u, v] : x \leq M x\}$ be the greatest fixed point of $M$. Combining the equations $\bar{x}_i = M i \bar{x}$ and $g_i^x(M i \bar{x})$, we get $\bar{x}_i = g_i^x(\bar{x}) = f_i(\bar{x})$ for each $i \in I$. Therefore $\bar{x}$ is also a fixed point of $f$. Since $x \leq f(x)$ implies $x \leq M x$ by (5), we claim $x \leq \bar{x}$ and hence $\bar{x} = \sup\{x \in [u, v] : x \leq f(x)\}$.

In a similar way we can prove that the smallest fixed point $\bar{x}$ of the increasing map $m : [u, v] \to [u, v]$ is the smallest fixed point of $f$ and that the second equation in (4) is fulfilled.

Corollary. Let $u$, $v \in X$, $u \leq v$, and $f = (f_i)_{i \in I} : [u, v] \to X$ be a function with the following properties:

- $f_i(x) \geq f_i(y)$ for $x$, $y \in [u, v]$ with $x \leq y$ and $x_i = y_i$, $i \in I$;
- $\max\{D_-f_i(x), D_+f_i(x)\} < \infty$ for $x \in [u, v]$, $i \in I$;
- $u_i \leq f_i(x + (u_i - x_i)e^i)$, $v_i \leq f_i(x + (v_i - x_i)e^i)$ for $x \in [u, v]$, $i \in I$.

Then $f$ has a greatest fixed point $\bar{x}$ and a smallest fixed point $\bar{x}$ and

$$\bar{x} = \sup\{x \in [u, v] : x \geq f(x)\}, \quad \bar{x} = \inf\{x \in [u, v] : x \leq f(x)\}.$$ 

Remarks. (1) In infinite-dimensional Banach spaces $X$ a function $f : X \to X$ is quasimonotone increasing if $x$, $y \in X$, $x \leq y$, $\varphi \in K^*$, $f(x) = \varphi(y)$ implies $\varphi(f(x)) \leq \varphi(f(y))$, where $K^* = \{\varphi \in X^* : \varphi(x) \geq 0 \text{ for all } x \in K\}$ (see Volkmann [3]). For $X = \mathbb{R}^n$, $c_0$, or $l^p$, $1 \leq p < \infty$, condition (1) is the same as quasimonotonicity. In case $X = l^\infty$ condition (1), even together with (2), is weaker than quasimonotonicity. Choose $\varphi \in K^*$ with $\varphi(x) = 0$ for $x \in c_0$ and $\varphi(e) = 1$ for $e = (e_n)_{n \in \mathbb{N}}$ with $e_n = 1$ for all $n \in \mathbb{N}$. For $x = (x_n)_{n \in \mathbb{N}} \in l^\infty$, define

$$f_n(x) = \begin{cases} 
1 & \text{for } x_n \leq 0, \\
1 - nx_n & \text{for } 0 \leq x_n \leq \frac{1}{n}, \\
0 & \text{for } x_n \leq \frac{1}{n},
\end{cases}$$

Then $f = (f_n)_{n \in \mathbb{N}} : l^\infty \to l^\infty$ satisfies (1) and (3), but is not quasimonotone increasing.

(2) In $X = l^\infty$, methods analogous to those used in this paper also lead to existence theorems for ordinary differential equations with quasimonotone right-hand side (see [1]).
References


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