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STABLE MEASURE OF A SMALL BALL

M. LEWANDOWSKI, M. RYZNAR, AND T. ŻAK

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ABSTRACT. Let μ be a symmetric p-stable measure on a Banach space $(E, \|\cdot\|)$. We prove that $\mu\{\|x\| < t\} \leq Kt$, where the constant K is independent of all properties of μ except for the measure of the unit ball $\mu\{\|x\| < 1\}$.

1. INTRODUCTION AND PRELIMINARY FACTS

Let μ be a symmetric Gaussian measure on a separable Banach space $(E, \|\cdot\|)$. In the paper of Szarek [7] there is the following bound for the distribution function of the norm

$$\mu(B_t) \leq Kt, \qquad t > 0, \tag{1}$$

where B_t is the ball with radius t and K is a constant depending only on $\mu(B_1)$. This inequality was used to obtain some results in the theory of computational complexity [7].

A similar result in the Hilbert space case was proved by Sawa [6]

$$\mu(B_t) \leq \Phi(t), \qquad 0 < t \leq 0,2$$

provided $\int \|x\|^2 \, d\mu(x) = 1$ and $\Phi(t) = \sqrt{\frac{2}{\pi}} \int_0^t \exp(-x^2/2)\, dx$.

The purpose of this note is both to extend (1) to the case of symmetric p-stable measures, $0 < p \leq 2$, and to give another proof for (1). Our method consists in using a series representation of stable random vectors obtained in [4].

Let us recall that a symmetric measure μ is called p-stable, $0 < p \leq 2$, iff for every t, $s > 0$ we have

$$tX + sY \overset{\mathrm{d}}{=} (s^p + t^p)^{1/p} X, \tag{2}$$

where X, Y are i.i.d. random vectors with the distribution μ. It is well known (see, e.g., [5]) that there exists a finite measure σ on $S_1 = \{x : \|x\| = 1\}$ such that the characteristic functional of μ has the form

$$\hat{\mu}(x^*) = \exp\left(-\int |x^*(x)|^p \sigma(dx)\right), \qquad x^* \in E^*.$$

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The measure σ is called the spectral measure of μ.

In the following lemmas we recall a series representation of p-stable random vectors in E. Let $(\alpha_i)_{i=1}^{\infty}$ and $(z_i)_{i=1}^{\infty}$ be two sequences of i.i.d. random variables such that $P(\alpha_1 > t) = e^{-t}$, $E|z_1|^p = 1$, and z_1 is symmetric. We assume that (α_i) and (z_i) are independent. Next, we denote $\Gamma_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$c_p = \left(\int_0^{\infty} v^{-p} \sin v \, dv\right)^{-1/p} = \left(\frac{1-p}{\Gamma(2-p)\cos \pi \frac{p}{2}}\right)^{1/p}.$$

Lemma 1 [4]. (a) *For $0 < p < 2$ the series $c_p \sum_{i=1}^{\infty} \Gamma_i^{-1/p} z_i$ is convergent a.s. and the characteristic function of the limit is* $\exp(-|t|^p)$.

(b) *For $0 < p < 1$ the series $\sum_{i=1}^{\infty} \Gamma_i^{-1/p}$ is convergent a.s. and has p-stable distribution with the characteristic function*

$$\exp(-c_p|t|^p(1 - i(\operatorname{sgn} t)\operatorname{tg} \pi \tfrac{p}{2})).$$

Lemma 2 [4]. *Let μ be a symmetric p-stable measure, $0 < p < 2$, with the spectral measure σ. Let $(V_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random vectors with the distribution $\sigma/\sigma(S_1)$, independent of (α_i) and (z_i). Then the series*

$$c_p[\sigma(S_1)]^{1/p} \sum_{i=1}^{\infty} \Gamma_i^{-1/p} V_i z_i \tag{3}$$

is convergent a.s. and has the distribution μ.

2. The main result

Theorem. *Suppose that μ is a symmetric p-stable measure, $0 < p \le 2$, such that $\int \|x\|^r \mu(d\mu) = 1$ for some $r \in (0, p)$. There exists a constant $K(p, r)$ depending only on p and r so that*

$$\mu(B_t) \le K(p, r)t, \qquad t > 0.$$

The proof consists of two steps. In the first one we find the appropriate estimate for $\mu(B_t)$ provided $\sigma(S_1) = 1$.

Step 1. We use Lemma 2 assuming that $(z_i)_{i=1}^{\infty}$ is a Gaussian sequence and $\sigma(S_1) = 1$. Suppose that (z_i) is defined on a probability space (Ω_1, P_1) and (α_i), (V_i) are defined on (Ω_2, P_2). When we fix (α_i) and (V_i) then the random vector $c_p \sum_{i=1}^{\infty} \Gamma_i^{-1/p} V_i z_i(\omega_1)$ is Gaussian. By Anderson's inequality [1] we have:

$$\begin{aligned}
&P_1\left(\left\|c_p \sum_{i=1}^{\infty} \Gamma^{-1/p} V_i z_i(\omega_1)\right\| \le t\right) \\
&\quad = P_1\left(\left\|c_p \Gamma_1^{-1/p} V_1 z_1(\omega_1) + \sum_{i=2}^{\infty} c_p \Gamma_i^{-1/p} V_i z_i(\omega_1)\right\| \le t\right) \\
&\quad \le P_1(\|c_p \Gamma_1^{-1/p} V_1 z_1(\omega_1)\| \le t) \\
&\quad = P_1(|z_1| \le c_p^{-1} \Gamma_1^{1/p} t) \qquad P_2 - \text{a.s.}
\end{aligned}$$

since $\|V_1\| = 1$ P_2-a.s. Therefore, by Fubini theorem

$$\mu(B_t) \le P_1 \times P_2(|z_1(\omega_1)| \le \Gamma_1^{1/p}(\omega_2) c_p^{-1} t) = \Psi_p(t), \tag{4}$$

where Ψ_p is the distribution function of $|z_1|\Gamma_1^{-1/p}c_p$.

Step 2. In this step we prove the theorem for p-stable measure μ, $0 < p \leq 2$, provided $\int \|x\|^r \, d\mu(x) = 1$, $r < p$. Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random vectors independent of (α_i) and with the distribution of X_1 equal to μ. Put

$$Y = c_r \sum_{i=1}^{\infty} \Gamma_i^{-1/r} X_i.$$

By Lemma 1 the characteristic functional of Y is equal to

$$\begin{aligned}\exp - \int |x^*(x)|^r \mu(dx) &= \exp - \int \left| x^* \left(\frac{x}{\|x\|} \right) \right|^r \|x\|^r \mu(dx) \\ &= \exp - \int |x^*(x)|^r \sigma_Y(dx),\end{aligned}$$

$$\text{where } \sigma_Y(S_1) = \int \|x\|^r \mu(dx) = 1.$$

Hence Y is an r-stable symmetric random vector with the spectral measure σ_Y so that by Step 1, $P(\|Y\| < t) \leq \Psi_r(t)$. On the other hand by the property (2) and the independence of (Γ_i) and (X_i) we obtain

$$Y \stackrel{\mathrm{d}}{=} c_r \left(\sum_{i=1}^{\infty} \Gamma_i^{-p/r} \right)^{1/p} X_1.$$

The series $\sum_{i=1}^{\infty} \Gamma_i^{-p/r} = \eta(\frac{r}{p})$ is convergent a.s. by Lemma 1 and $\eta(\frac{r}{p})$ is independent of X_1. Therefore, for every $a > 0$:

$$\begin{aligned}\Psi_r(at) \geq P(\|Y\| \leq at) &= P(c_r \left[\eta \left(\frac{r}{p} \right) \right]^{1/p} \|X_1\| \leq at) \\ &\geq P(c_r \eta^{1/p} \leq a, \ \|X_1\| \leq t) = P\left(\eta \leq \left(\frac{a}{c_r} \right)^p \right) \cdot P(\|X_1\| \leq t).\end{aligned}$$

Hence

$$\mu(B_t) \leq \frac{\Psi_r(at)}{P(\eta \leq (\frac{a}{c_r})^p)}. \tag{5}$$

Simple calculations give $\sup_{t>0} \Psi_r'(t) = C < \infty$. Putting

$$K(p, r) = \sup_{t>0} \Psi_r'(t) \cdot \inf_{a>0} a \left[P\left(\eta \leq \left(\frac{a}{c_r} \right)^p \right) \right]^{-1}$$

we get the desired conclusion.

For arbitrary values of $\int \|x\|^r \mu(dx)$ we obtain, by the Čebyshev inequality,

$$\begin{aligned}\mu(B_t) &\leq K(p, r) \left[\int \|x\|^r \mu(dx) \right]^{-1/r} \cdot t \\ &\leq K(p, r)[1 - \mu(B_1)]^{-1/r} \cdot t.\end{aligned}$$

This inequality is a version of (1) for stable case $p < 2$ and for $p = 2$ it is precisely (1).

3. Estimation of constants $K(p, r)$

First we estimate the value of $\sup_{t>0} \Psi_r'(t)$. Recall $\Psi_r(t) = P(|z_1|\Gamma_1^{-1/r}c_r \leq t)$, z_1 has the distribution $N(0, \sigma_r)$, where $\sigma_r = \pi^{1/2r}(\sqrt{2}\Gamma^{1/r}(\frac{r+1}{2}))^{-1}$, $c_r =$

$((1-r)/\Gamma(2-r)\cos\pi\frac{r}{2})^{1/r}$, and $c_1 = \frac{2}{\pi}$.

Now

$$\begin{aligned}\Psi_r'(t) &= \int_0^\infty \frac{d}{dt} P\left(|z_1| \le \frac{t}{c_r} x^{1/r}\right) e^{-x}\,dx \\ &= \int_0^\infty \frac{\sqrt{2}}{\sqrt{\pi}\sigma_r} \exp\left(-\frac{x^{2/r}t^2}{2c_r^2\sigma_r^2}\right) \frac{x^{1/r}}{c_r} e^{-x}\,dx \\ &\le \sqrt{\frac{2}{\pi}} \frac{\Gamma(1+1/r)}{c_r\sigma_r} = A_r.\end{aligned}$$

We will also need an evaluation of the distribution function of $\eta(\frac{1}{2}) = \sum_{i=1}^\infty \Gamma_i^{-2}$ which, by Lemma 1(b), has $\frac{1}{2}$-stable distribution on R^+ with the density $f(x) = 1/(2x^{3/2})\exp(-\pi/(4x))$ for $x > 0$. It is easy to notice that $P(\eta(\frac{1}{2}) < t) = 1 - \Phi(\sqrt{\frac{\pi}{2t}})$.

(a) $p = 2$, $r = 1$. This is the Gaussian case. Assume that $\int \|x\|\mu(dx) = 1$. By (5),

$$\begin{aligned}\mu(B_t) &\le A_1 \cdot \inf_{a>0} a \left[P\left(\eta\left(\frac{1}{2}\right) \le \left(\frac{a}{c_1}\right)^2\right)\right]^{-1} t \\ &= A_1\sqrt{\frac{2}{\pi}} c_1 \left[\sup_{r>0} r(1-\Phi(r))\right]^{-1} t \le 2{,}35t,\end{aligned}$$

because $A_1 = 1$, $c_1 = \frac{2}{\pi}$, and the above supremum is attained in a neighborhood of $r = 0{,}75$.

(b) For arbitrary $p \in (0, 2)$ we take $r = \frac{p}{2}$ (because in this case we know the density of $\eta(\frac{r}{p})$). In this case

$$\begin{aligned}(6)\quad K\left(p, \frac{p}{2}\right) &= A_{p/2} \inf_{a>0} a \left[P\left(\eta\left(\frac{1}{2}\right) \le \left(\frac{a}{c_{p/2}}\right)^p\right)\right]^{-1} \\ &= \frac{1}{2^{1/p}\sqrt{\pi}} \Gamma^{2/p}(p/4 + 1/2)\Gamma(1 + 2/p) \inf_{r>0} (r^{2/p}(1-\Phi(r)))^{-1}.\end{aligned}$$

Observe $K(p, \frac{p}{2})$ tends to infinity very rapidly as $p \to 0$. But, when $p \ge \varepsilon > 0$ we can find some upper bound for it. For example, if $1 \le p < 2$ then the properties of the function $\Gamma(x)$ give the estimate $\mu(B_t) \le 2\sqrt{2\pi}/(1-\Phi(1)) \cdot t \le 15{,}8t$, when we take $r = 1$ in (6) for simplicity.

(c) If $0 < p < 1$ we can give an estimate better than (6).

It is well known that in Banach spaces of stable type p there holds an inequality between the rth moment of a p-stable measure and the total mass of its spectral measure, and every Banach space is of stable type p, $p < 1$ (see, e.g., [5]). Namely, as it was shown by Pisier [2, Lemma 5.4]: if X is a p-stable random vector, $0 < p < 1$, $0 < r < p$, and σ is its spectral measure then

$$(7)\qquad [E\|X\|^r]^{1/r} \le \frac{c_p(r)c_1(p)}{c_1(r)} [\sigma(S_1)]^{1/p},$$

where $c_p^r(r) = 2^{r-1}\Gamma(1-r/p)(r\int_0^\infty u^{-r-1}\sin^2 u\,du)^{-1}$ denotes the rth moment of the standard symmetric p-stable random variable on R (for the value of

$c_p(r)$ compare [3]). But

$$r\int_0^\infty u^{-r-1}\sin^2 u\,du = 2^{r-1}\int_0^\infty u^{-r}\sin u\,du = 2^{r-1}c_r^{-r}$$
$$= 2^{r-1}\Gamma(2-r)\cos\frac{\pi r}{2}\cdot[(1-r)]^{-1}.$$

Finally

$$c_p(r) = \left[\frac{\Gamma(1-r/p)(1-r)}{\Gamma(2-r)\cos\frac{\pi r}{2}}\right]^{1/r}$$

and, by (7),

$$(E\|X\|^r)^{1/r} \le \left[\frac{\Gamma(1-r/p)}{\Gamma(1-r)}\right]^{1/r}\left[\frac{\Gamma(1-p)(1-p)}{\Gamma(2-p)\cos\frac{\pi r}{2}}\right]^{1/p}\cdot[\sigma(S_1)]^{1/p}$$
$$= B_p(r)[\sigma(S_1)]^{1/p}.$$

We use only Step 1. If μ is symmetric p-stable with the spectral measure σ then by (4) and (7) and the definitions of A_p and $B_p(r)$:

$$\mu(B_t) \le \frac{\Psi_p(t)}{[\sigma(S_1)]^{1/p}}$$
$$\le \frac{\Psi_p(t)B_p(r)}{\left[\int\|x\|^r\mu(dx)\right]^{1/r}} \le \frac{A_pB_p(r)}{\left[\int\|x\|^r\mu(dx)\right]^{1/r}}\cdot t.$$

Remarks. (1) If p is a concrete given number we can estimate more carefully in (6) and get better constant.

(2) For symmetric p-stable random variable on R with characteristic function $\exp(-|t|^p)$ we have $P(|X|<t)\le\pi^{-1}\Gamma(1+1/p)t$ because $\Gamma(1+1/p)/\pi$ is precisely the value of the density at zero, hence the constant $A_pB_p(r)$ must tend to infinity at least like $\Gamma(1+1/p)$ when $p\to 0$.

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Added in proof. When the first draft of this paper was circulating, the second author had a talk on this subject during the Banach Center Probability Workshop (June 1990). After his talk Professor X. Fernique kindly informed us that for the Gaussian case ($p=2$) he knew another two proofs. One is explicitly contained in his paper *Les vecteurs aléatoires gaussiens et leurs espaces autoreproduisants*, Technical Report 34, Ser. Lab. Res. Statist. and Probab., University of Ottawa, 1985.

For the second, one can use an inequality of Kanter (*Probability inequalities for convex sets and multidimensional concentration functions*, J. Multivariate Anal. **6** (1976), 222–236, inequality 4.1): for $(X_i)_{i=1}^n$ independent and symmetric

$$P\left(\left\|\sum_{i=1}^n X_i\right\|<t\right)\le\left(\frac{3}{2}\right)\left(1+\sum_{i=1}^n P(\|X_i\|>t)\right)^{-1/2}.$$

Taking $X_1 \overset{d}{=} X_2 \overset{d}{=} \cdots \overset{d}{=} X/\sqrt{n}$ and $n = [\frac{1}{t^2}]$ we get the desired conclusion.

Fernique's proof is based on the rotational invariance of the product of Gaussian measures; hence it does not immediately apply for $p < 2$. The second method gives the estimate of order $t^{p/2}$, which is worse than t for $t \to 0$, if $p < 2$.

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Technical University, Department of Mathematics, Wybrzeże Wyspianskiego 27, Wroclaw 50 370, Poland