STABLE MEASURE OF A SMALL BALL

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Abstract. Let \( \mu \) be a symmetric \( p \)-stable measure on a Banach space \((E, \| \cdot \|)\). We prove that \( \mu(\|x\| < t) \leq Kt \), where the constant \( K \) is independent of all properties of \( \mu \) except for the measure of the unit ball \( \mu(\|x\| < 1) \).

1. Introduction and preliminary facts

Let \( \mu \) be a symmetric Gaussian measure on a separable Banach space \((E, \| \cdot \|)\). In the paper of Szarek [7] there is the following bound for the distribution function of the norm

\[ \mu(B_t) \leq K t, \quad t > 0, \]

where \( B_t \) is the ball with radius \( t \) and \( K \) is a constant depending only on \( \mu(B_1) \). This inequality was used to obtain some results in the theory of computational complexity [7].

A similar result in the Hilbert space case was proved by Sawa [6]

\[ \mu(B_t) \leq \Phi(t), \quad 0 < t \leq 0, 2 \]

provided \( \int \|x\|^2 d\mu(x) = 1 \) and \( \Phi(t) = \sqrt{\frac{2}{\pi}} \int_0^t \exp(-x^2/2) \, dx \).

The purpose of this note is both to extend (1) to the case of symmetric \( p \)-stable measures, \( 0 < p \leq 2 \), and to give another proof for (1). Our method consists in using a series representation of stable random vectors obtained in [4].

Let us recall that a symmetric measure \( \mu \) is called \( p \)-stable, \( 0 < p \leq 2 \), iff for every \( t, s > 0 \) we have

\[ tX + sY \overset{d}{=} (s^p + t^p)^{1/p} X, \]

where \( X, Y \) are i.i.d. random vectors with the distribution \( \mu \). It is well known (see, e.g., [5]) that there exists a finite measure \( \sigma \) on \( S_1 = \{x : \|x\| = 1\} \) such that the characteristic functional of \( \mu \) has the form

\[ \hat{\mu}(x^*) = \exp\left(-\int |x^*(x)|^p \sigma(dx)\right), \quad x^* \in E^*. \]
The measure $\sigma$ is called the spectral measure of $\mu$.

In the following lemmas we recall a series representation of $p$-stable random vectors in $E$. Let $(\alpha_i)_{i=1}^\infty$ and $(z_i)_{i=1}^\infty$ be two sequences of i.i.d. random variables such that $P(\alpha_1 > t) = e^{-t}$, $E|z_1|^p = 1$, and $z_1$ is symmetric. We assume that $(\alpha_i)$ and $(z_i)$ are independent. Next, we denote $\Gamma_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$c_p = \left( \int_0^\infty v^{-p} \sin v \, dv \right) -1/p = \left( \frac{1-p}{\Gamma(2-p)} \right)^{1/p}.$$

**Lemma 1** [4]. (a) For $0 < p < 2$ the series $c_p \sum_{i=1}^\infty \Gamma_i^{-1/p} z_i$ is convergent a.s. and the characteristic function of the limit is $\exp(-|t|^p)$.

(b) For $0 < p < 1$ the series $\sum_{i=1}^\infty \Gamma_i^{-1/p}$ is convergent a.s. and has $p$-stable distribution with the characteristic function

$$\exp(-c_p |t|^p (1 - i \text{sgn}(t) \tan \pi \frac{p}{2})).$$

**Lemma 2** [4]. Let $\mu$ be a symmetric $p$-stable measure, $0 < p < 2$, with the spectral measure $\sigma$. Let $(V_i)_{i=1}^\infty$ be a sequence of i.i.d. random vectors with the distribution $\sigma/\sigma(S_1)$, independent of $(\alpha_i)$ and $(z_i)$. Then the series

$$c_p \sum_{i=1}^\infty \Gamma_i^{-1/p} V_i z_i$$

is convergent a.s. and has the distribution $\mu$.

2. The main result

**Theorem.** Suppose that $\mu$ is a symmetric $p$-stable measure, $0 < p < 2$, such that $\int \|x\|^r d\mu(x) = 1$ for some $r \in (0, p)$. There exists a constant $K(p, r)$ depending only on $p$ and $r$ so that

$$\mu(B_t) \leq K(p, r) t, \quad t > 0.$$

The proof consists of two steps. In the first one we find the appropriate estimate for $\mu(B_t)$ provided $\sigma(S_1) = 1$.

**Step 1.** We use Lemma 2 assuming that $(z_i)_{i=1}^\infty$ is a Gaussian sequence and $\sigma(S_1) = 1$. Suppose that $(z_i)$ is defined on a probability space $(\Omega_1, P_1)$ and $(\alpha_i), (V_i)$ are defined on $(\Omega_2, P_2)$. When we fix $(\alpha_i)$ and $(V_i)$ then the random vector $c_p \sum_{i=1}^\infty \Gamma_i^{-1/p} V_i z_i(\omega_1)$ is Gaussian. By Anderson’s inequality [1] we have:

$$P_1 \left( \left\| c_p \sum_{i=1}^\infty \Gamma_i^{-1/p} V_i z_i(\omega_1) \right\| \leq t \right)$$

$$= P_1 \left( \left\| c_p \Gamma_1^{-1/p} V_1(\omega_1) + \sum_{i=2}^\infty c_p \Gamma_i^{-1/p} V_i (\omega_1) \right\| \leq t \right)$$

$$\leq P_1 (\| c_p \Gamma_1^{-1/p} V_1 z_i(\omega_1) \| \leq t)$$

$$= P_1 (|z_1| \leq c_p^{-1} \Gamma_1^{1/p} t) \quad P_2 - \text{a.s.}$$

since $\|V_1\| = 1$. $P_2$-a.s. Therefore, by Fubini theorem

$$\mu(B_t) \leq P_1 \times P_2 (|z_1(\omega_1)| \leq \Gamma_1^{1/p} (\omega_2) c_p^{-1} t) = \Psi_p(t),$$

where $\Psi_p(t)$ is a function of $t$. Therefore, we obtain the estimate $\mu(B_t) \leq K(p, r) t$. The proof is completed.
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where \( \Psi_p \) is the distribution function of \( |z_1| \Gamma_1^{-1/p} c_p \).

**Step 2.** In this step we prove the theorem for \( p \)-stable measure \( \mu \), \( 0 < p < 2 \), provided \( \int \|x\|^r d\mu(x) = 1 \), \( r < p \). Let \( (X_i)_{i=1}^{\infty} \) be a sequence of i.i.d. random vectors independent of \((\alpha_i)\) and with the distribution of \( X_1 \) equal to \( \mu \). Put

\[
Y = c_r \sum_{i=1}^{\infty} \Gamma_i^{-1/r} X_i.
\]

By Lemma 1 the characteristic functional of \( Y \) is equal to

\[
\exp - \int |x^*(x)|^r \mu(dx) = \exp - \int \left| x^* \left( \frac{x}{\|x\|} \right) \right|^r \|x\|^r \mu(dx)
\]

\[
= \exp - \int |x^*(x)|^r \sigma_Y(dx),
\]

where \( \sigma_Y(S_1) = \int \|x\|^r \mu(dx) = 1 \).

Hence \( Y \) is an \( r \)-stable symmetric random vector with the spectral measure \( \sigma_Y \) so that by Step 1, \( P(\|Y\| < t) \leq \Psi_r(t) \). On the other hand by the property (2) and the independence of \((\Gamma_i)\) and \((X_i)\) we obtain

\[
\sum_{i=1}^{\infty} \Gamma_i^{-p/r} = \eta\left( \frac{r}{p} \right) \text{ is convergent a.s. by Lemma 1 and } \eta\left( \frac{r}{p} \right) \text{ is independent of } X_1.\]

Therefore, for every \( a > 0 \):

\[
\Psi_r(at) \geq P(\|Y\| \leq at) = P(c_r \left[ \eta \left( \frac{r}{p} \right) \right]^{1/p} \|X_1\| \leq at)
\]

\[
\geq P(c_r \eta^{1/p} \leq a, \|X_1\| \leq t) = P \left( \eta \leq \left( \frac{a}{c_r} \right)^p \right) \cdot P(\|X_1\| \leq t).
\]

Hence

\[
\mu(B_t) \leq \frac{\Psi_r(at)}{P(\eta \leq \left( \frac{a}{c_r} \right)^p)}. \tag{5}
\]

Simple calculations give \( \sup_{t>0} \Psi_r(t) = C < \infty \). Putting

\[
K(p, r) = \sup_{t>0} \Psi_r(t) \cdot \inf_{a>0} a \left[ P \left( \eta \leq \left( \frac{a}{c_r} \right)^p \right) \right]^{-1}
\]

we get the desired conclusion.

For arbitrary values of \( \int \|x\|^r \mu(dx) \) we obtain, by the Čebyshev inequality,

\[
\mu(B_t) \leq K(p, r) \left[ \int \|x\|^r \mu(dx) \right]^{-1/r} \cdot t
\]

\[
\leq K(p, r) \left[ 1 - \mu(B_1) \right]^{-1/r} \cdot t.
\]

This inequality is a version of (1) for stable case \( p < 2 \) and for \( p = 2 \) it is precisely (1).

3. Estimation of constants \( K(p, r) \)

First we estimate the value of \( \sup_{t>0} \Psi_r(t) \). Recall \( \Psi_r(t) = P(|z_1| \Gamma_1^{-1/r} c_r \leq t) \), \( z_1 \) has the distribution \( N(0, \sigma_r) \), where \( \sigma_r = \pi^{1/2r} (2 \Gamma^{1/r}(\frac{r+1}{2}))^{-1} \), \( c_r = \ldots \)
Now
\[ \Psi_r(t) = \int_0^\infty \frac{d}{dt} P \left( |z_1| \leq \frac{t}{c_1} r^{1/r} \right) e^{-x} \, dx \]
\[ = \int_0^\infty \frac{\sqrt{2}}{\sqrt{\pi} \sigma_r} \exp \left( -\frac{x^2}{2 \sigma_r^2} \right) \frac{x^{1/r}}{c_r} e^{-x} \, dx \]
\[ \leq \sqrt{\frac{2}{\pi}} \frac{\Gamma(1 + 1/r)}{c_r \sigma_r} = A_r. \]

We will also need an evaluation of the distribution function of \( \eta(\frac{1}{2}) = \sum_{i=1}^\infty \Gamma_i^{-2} \) which, by Lemma 1(b), has \( \frac{1}{2} \)-stable distribution on \( \mathbb{R}^+ \) with the density \( f(x) = 1/(2x^{3/2}) \exp(-\pi/(4x)) \) for \( x > 0 \). It is easy to notice that \( P(\eta(\frac{1}{2}) < t) = 1 - \Phi(\frac{\sqrt{t}}{2}) \).

(a) \( p = 2, r = 1 \). This is the Gaussian case. Assume that \( \int \|x\| \mu(dx) = 1 \).

By (5),
\[ \mu(B_1) \leq A_1 \cdot \inf_{a > 0} \left[ P \left( \eta \left( \frac{1}{2} \right) \leq \left( \frac{a}{c_1} \right)^2 \right) \right]^{-1} \]
\[ = A_1 \sqrt{\frac{2}{\pi}} c_1 \left[ \sup_{r > 0} r(1 - \Phi(r)) \right]^{-1} \]
\[ \leq 2, 35t, \]
because \( A_1 = 1, c_1 = \frac{2}{\pi} \), and the above supremum is attained in a neighborhood of \( r = 0, 75 \).

(b) For arbitrary \( p \in (0, 2) \) we take \( r = \frac{p}{2} \) (because in this case we know the density of \( \eta(\frac{p}{2}) \)). In this case
\[ (6) \quad K \left( p, \frac{p}{2} \right) = A_{p/2} \inf_{a > 0} \left[ P \left( \eta \left( \frac{1}{2} \right) \leq \left( \frac{a}{c_{p/2}} \right)^p \right) \right]^{-1} \]
\[ = \frac{1}{2^{1/p}} \sqrt{\pi} \Gamma^{2/p}(p/4 + 1/2) \Gamma(1 + 2/p) \inf_{r > 0} r^{2/p}(1 - \Phi(r))^{-1}. \]

Observe \( K(p, \frac{p}{2}) \) tends to infinity very rapidly as \( p \to 0 \). But, when \( p \geq e > 0 \) we can find some upper bound for it. For example, if \( 1 \leq p < 2 \) then the properties of the function \( \Gamma(x) \) give the estimate \( \mu(B_1) \leq 2\sqrt{2\pi}/(1 - \Phi(1)) \cdot t \leq 15, 8t \), when we take \( r = 1 \) in (6) for simplicity.

(c) If \( 0 < p < 1 \) we can give an estimate better than (6).

It is well known that in Banach spaces of stable type \( p \) there holds an inequality between the \( r \) th moment of a \( p \)-stable measure and the total mass of its spectral measure, and every Banach space is of stable type \( p, p < 1 \) (see, e.g., [5]). Namely, as it was shown by Pisier [2, Lemma 5.4]: if \( X \) is a \( p \)-stable random vector, \( 0 < p < 1 \), \( 0 < r < p \), and \( \sigma \) is its spectral measure then
\[ (7) \quad [E\|X\|']^{1/r} \leq \frac{c_p(r)c_1(p)}{c_1(r)} [\sigma(S_1)]^{1/p}, \]
where \( c_p(r) = 2r^{-1} \Gamma(1 - r/p)(\int_0^\infty u^{-r-1} \sin^2 u \, du)^{-1} \) denotes the \( r \) th moment of the standard symmetric \( p \)-stable random variable on \( \mathbb{R} \) (for the value of
$c_p(r)$ compare [3]). But

$$r \int_0^\infty u^{-r-1} \sin^2 u \, du = 2^{r-1} \int_0^\infty u^{-r} \sin u \, du = 2^{r-1} c_r^{-r}$$

$$= 2^{r-1} \Gamma(2 - r) \cos \frac{\pi r}{2} \cdot [(1 - r)]^{1/r}.$$

Finally

$$c_p(r) = \left[ \frac{\Gamma(1 - r/p)(1 - r)}{\Gamma(2 - r) \cos \frac{\pi r}{2}} \right]^{1/r}$$

and, by (7),

$$(E\|X\|^r)^{1/r} \leq \left[ \frac{\Gamma(1 - r/p)}{\Gamma(1 - r)} \right]^{1/r} \left[ \frac{\Gamma(1 - p)(1 - p)}{\Gamma(2 - p) \cos \frac{\pi r}{2}} \right]^{1/p} \cdot [\sigma(S_1)]^{1/p}$$

$$= B_p(r)[\sigma(S_1)]^{1/p}.$$

We use only Step 1. If $\mu$ is symmetric $p$-stable with the spectral measure $\sigma$ then by (4) and (7) and the definitions of $A_p$ and $B_p(r)$:

$$\mu(B_t) \leq \frac{\Psi_p(t)}{[\sigma(S_1)]^{1/p}} \leq \frac{\Psi_p(t) B_p(r)}{[\int \|x\|^{1/r} \mu(dx)]^{1/r}} \leq A_p B_p(r) \cdot t.$$

**Remarks.** (1) If $p$ is a concrete given number we can estimate more carefully in (6) and get better constant.

(2) For symmetric $p$-stable random variable on $R$ with characteristic function $\exp(-|t|^p)$ we have $P(|X| < t) \leq \pi^{-1} \Gamma(1 + 1/p)t$ because $\Gamma(1 + 1/p)/\pi$ is precisely the value of the density at zero, hence the constant $A_p B_p(r)$ must tend to infinity at least like $\Gamma(1 + 1/p)$ when $p \to 0$.

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*Added in proof.* When the first draft of this paper was circulating, the second author had a talk on this subject during the Banach Center Probability Workshop (June 1990). After his talk Professor X. Fernique kindly informed us that for the Gaussian case ($p = 2$) he knew another two proofs. One is explicitly contained in his paper *Les vecteurs aléatoires gaussiens et leurs espaces autoreproduisants*, Technical Report 34, Ser. Lab. Res. Statist. and Probab., University of Ottawa, 1985.

For the second, one can use an inequality of Kanter (*Probability inequalities for convex sets and multidimensional concentration functions*, J. Multivariate Anal. 6 (1976), 222–236, inequality 4.1): for $(X_i)_{i=1}^n$ independent and symmetric

$$P \left( \left\| \sum_{i=1}^n X_i \right\| < t \right) \leq \left( \frac{3}{2} \right) \left( 1 + \sum_{i=1}^n P (\|X_i\| > t) \right)^{-1/2}.$$
Taking $X_1 \overset{d}{=} X_2 \overset{d}{=} \cdots \overset{d}{=} X/\sqrt{n}$ and $n = [\frac{1}{t}]$ we get the desired conclusion.

Fernique's proof is based on the rotational invariance of the product of Gaussian measures; hence it does not immediately apply for $p < 2$. The second method gives the estimate of order $t^{p/2}$, which is worse than $t$ for $t \to 0$, if $p < 2$.

REFERENCES


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