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# SOME REMARKS OF DROP PROPERTY

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ABSTRACT. Let $C$ be a proper closed convex set. $C$ is said to have the drop property if for any nonempty closed set $A$ disjoint with $C$, there is $a \in A$ such that $\mathrm{co}(a, C) \cap A = \{a\}$. We show that if $X$ contains a noncompact set with the drop property, then $X$ is reflexive. Moreover, we prove that if $C$ is a noncompact closed convex subset of a reflexive Banach space, then $C$ has the drop property if and only if $C$ satisfies the following conditions: (i) the interior of $C$ is nonempty; (ii) $C$ does not have any asymptote, and the boundary of $C$ does not contain any ray; and (iii) every support point $x$ of $C$ is a point of continuity.

## 1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space, and let $C$ be a nonempty proper closed convex subset of $X$. For any $x \notin C$, the *drop* determined by $x$ is the set $D(x, C) = \mathrm{co}(x, C)$, the convex hull of the set $\{x\} \cup C$. Daneš [D] proved that if $C$ is a bounded closed subset of $X$ and $A$ is a closed set at positive distance from $C$, then there exists an $a \in A$ such that $D(a, C) \cap A = \{a\}$. Modifying the assumption, Rolewicz [R1] said a nonempty proper closed set $C$ has the *drop property* if for every nonempty closed set $A$ disjoint with $C$, there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. The bounded closed convex sets with the drop property are studied in [K1, K2, M, R1, R2]. In [R1] Rolewicz proved that if the closed unit ball of $X$ has the drop property (in this case, we say $X$ has the *drop property*), then $X$ is reflexive. Kutzarova [K1] extended this result by showing $X$ is reflexive if $X$ contains a noncompact bounded closed convex set (respectively, a noncompact balanced closed convex set) with the drop property. Recently, Kutzarova and Rolewicz [KR1] showed that $X$ is reflexive if $X$ contains a noncompact closed convex symmetric set with the drop property.

For any subset $C$ of $X$, the *Kuratowski measure* of $C$ is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of $C$ by a finite number of sets of diameter less then $\varepsilon$. It is known that $\alpha(C) = 0$ if and only if $C$ is totally bounded. Let $C$ be a closed convex subset of $X$. We denote the set of all nonzero linear functionals $f \in X^*$, which are bounded above $C$ by $F(C)$.

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For any $f \in F(C)$, and any $\delta > 0$, the slice $S(f, C, \delta)$ is the set

$$\{x \in C : f(x) \geq M - \delta\},$$

where $M = \sup\{f(x) : x \in C\}$. A closed convex set $C$ is said to have *property* $(\alpha)$ if

$$\lim_{\delta \to 0} \alpha(S(f, C, \delta)) = 0$$

for all $f \in F(C)$. It is easy to see that a closed convex set $C$ has property $(\alpha)$ if and only if for any $f \in F(C)$ and $x_n \in S(f, C, \frac{1}{n})$, $\{x_n\}$ contains a convergent subsequence. In [KR1] Kutzarova and Rolewicz proved the following

**Theorem A.** *Let $C$ be any closed convex subset of $X$.*

(i) *If $C$ has the drop property, then $C$ has property $(\alpha)$.*

(ii) *If $C$ is not compact and if $C$ has the drop property, then $C$ has nonempty interior.*

(iii) *Suppose $X$ is reflexive. If $C$ has nonempty interior and $C$ has property $(\alpha)$, then $C$ has the drop property.*

(iv) *Let $C$ be a closed bounded convex set of a reflexive Banach space. If $\operatorname{int}(C) \neq \varnothing$ (where $\operatorname{int}(C)$ is the interior of $C$) and every support point of $C$ is a point of continuity, then $C$ has drop property.*

Using Theorem A, they proved that if $C_1$ and $C_2$ are any two bounded sets with the drop property, then $C_1 \cap C_2$, $C_1 + C_2$, and $\operatorname{co}(C_1, C_2)$ have the drop property. In §2 we show the assumption of boundedness can be removed. Hence, if $X$ contains a noncompact closed convex set with the drop property, then $X$ is reflexive. This gives an answer to a question of D. N. Kutzarova and S. Rolewicz [KR1].

Let $C$ be a closed convex set. $C$ is said to have *property* $(*)$ if $C$ contains the ray $\{c + \lambda b : \lambda \geq 0\}$ implies for any $x \in X$, there is $\beta > 0$ such that $x + (\beta + \lambda)b \in C$ for every $\lambda \geq 0$. In §2 we prove that if $C$ is a noncompact proper closed convex set of a reflexive Banach space, then $C$ has the drop property if and only if $\operatorname{int}(C) \neq \varnothing$, $C$ has property $(*)$, and every support point of $C$ is a point of continuity. This gives an extension of Theorem A(iv).

Recall a space $X$ is said to have the *Kadec-Klee property* (or *property* (H)) if on the unit sphere the weakly convergent sequence is convergent in norm (i.e., if $\|x_n\| = 1$ and $x_n$ converges weakly to a unit vector $x$, then $x_n$ converges to $x$ in norm). V. Montesinos [M] proved that $X$ has the drop property if and only if $X$ is reflexive and $X$ has the Kadec-Klee property. Recall that a sequence $\{x_n\}$ is said to be an *$\epsilon$-separate sequence* for some $\epsilon > 0$ if $\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon$. A Banach space $X$ is said to have the *uniform Kadec-Klee property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x$ is a weak limit of a norm one $\epsilon$-separate sequence, then $\|x\| < 1 - \delta$. A Banach space is said to be *nearly uniformly convex* (NUC) if for every $\epsilon > 0$ there exists a $\delta$, $1 > \delta > 0$, such that for every sequence $\{x_n\} \subseteq B$ with $\operatorname{sep}(x_n) > \epsilon$, we have $\operatorname{co}(x_n) \cap (1 - \delta)B \neq \varnothing$. It is easy to see that every (NUC) space has the uniform Kadec-Klee property, and every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. Huff [H] proved that $X$ is (NUC) if and only if $X$ is reflexive and $X$ has the uniform Kadec-Klee property. Modifying the theorem, Kutzarova and Rolewicz [KR2] said a closed convex set is (NUC) (respectively (NUC′)) with respect to a center $c \in C$ if for every $\epsilon > 0$ there exists a $\delta$, $1 > \delta > 0$, such that for every $\varepsilon$-separate sequence $\{x_n\} \subseteq C$

$$\operatorname{co}(x_n) \cap (1-\delta)(C-c) \neq \varnothing$$
$$(\text{respectively, } \overline{\operatorname{co}}(x_n) \cap (1-\delta)(C-c) \neq \varnothing).$$

It is easy to see that if $C$ is (NUC) with respect to $c \in C$, then $C$ is (NUC′) with respect to $c$. Kutzarova and Rolewicz [KR2] proved that if $c$ is an interior point of $C$, then $C$ is (NUC) with respect to $c$ if and only if $C$ is (NUC′) with respect to any $c$. They asked whether this is still true if $c$ is a boundary point of $C$. In §3, we show this is true if $C$ has the drop property. We also give an example to show the assumption of the drop property cannot be removed.

## 2. On the drop property

In [KR1] Kutzarova and Rolewicz asked whether $X$ is reflexive if $X$ contains a noncompact closed convex with the drop property. The following theorem shows the answer is affirmative.

**Theorem 1.** *Let $C_1$ and $C_2$ be any two closed convex subsets of $X$ with the drop property. If $C_1 \cap C_2 \neq \varnothing$, then $C_1 \cap C_2$ has the drop property. Hence, if $X$ contains a noncompact closed convex set with the drop property, then $X$ is reflexive.*

*Proof.* Let $A$ be any closed subset of $X$ such that $A \cap (C_1 \cap C_2) = \varnothing$. If $A \cap C_1 = \varnothing$, then there exists $a \in A$ such that $D(a, C_1) \cap A = \{a\}$. This implies $D(a, (C_1 \cap C_2)) \cap A = \{a\}$. So we may assume that $A \cap C_1 \neq \varnothing$. Since $(A \cap C_1) \cap C_2 = \varnothing$ and $C_2$ has the drop property, there is $a \in A \cap C_1$ such that $D(a, C_2) \cap (A \cap C_1) = \{a\}$. So

$$D(a, (C_2 \cap C_1)) \cap A \subseteq (D(a, (C_2 \cap C_1)) \cap C_1) \cap A = \{a\},$$

and $C_1 \cap C_2$ has the drop property.

It is easy to see that if $X$ contains a noncompact closed convex set with the drop property, then $X$ contains a noncompact symmetric closed convex set with the drop property. By [KR1, Proposition 4], $X$ is a reflexive space. □

*Remark* 1. Let $C$ be an unbounded closed convex set of a reflexive space. Kutzarova and Rolewicz proved that if $S(f, C, 1)$ is bounded for some $f \in F(C)$, then $C$ contains a ray $\{c + \beta b : \beta > 0\}$. Moreover, if $c' \in C$, $C$ also contains the ray $\{c' + \beta b : \beta > 0\}$.

Let $C_1$ and $C_2$ be any two bounded closed convex sets with the drop property. In [KR1] Kutzarova and Rolewicz proved that $\lambda C_1 + \mu C_2$ and $\operatorname{co}(C_1, C_2)$ have the drop property. The following theorem shows that the boundedness can be removed.

**Theorem 2.** *Let $C_1$ and $C_2$ be any two closed convex sets with the drop property. If $co(C_1, C_2) \neq X$, then*

(i) *for any $\lambda, \mu \neq 0$, $\lambda C_1 + \mu C_2$ is closed, and it has the drop property*;
(ii) *$co(C_1, C_2)$ is closed, and it has the drop property.*

*Proof.* We only prove (ii) and leave the proof of (i) to the reader. If $C_1$ and $C_2$ are compact, then $\operatorname{co}(C_1, C_2)$ is compact. So we may assume that $\operatorname{co}(C_1, C_2)$ has an interior point.

First, we show $\operatorname{co}(C_1, C_2)$ has property $(\alpha)$. If $f \in F(\operatorname{co}(C_1, C_2))$, then $f \in F(C_1) \cap F(C_2)$. Let $x$ (respectively, $x'$) be a point in $C_1$ (respectively, $C_2$) such that

$$f(x) = \sup\{f(y) : y \in C_1\}$$
$$(\text{respectively, } f(x') = \sup\{f(y) : y \in C_2\}).$$

One can easily show that

$$S(f, \operatorname{co}(C_1, C_2), \delta) = \operatorname{co}(x, x') + \big(S(f, C_1, \delta) - x\big) + \big(S(f, C_2, \delta) - x'\big).$$

(Compare with the proof of [KR1, Theorem 9 (iii)].) So

$$\lim_{\delta \to 0} \alpha(S(f, \operatorname{co}(C_1, C_2), \delta) = 0$$

and

$$\operatorname{co}(C_1, C_2) \text{has property } (\alpha).$$

Suppose that $b \in \overline{\operatorname{co}(C_1, C_2)} \setminus \operatorname{co}(C_1, C_2) \neq \varnothing$. By Hahn-Banach Theorem, there is a linear functional $f$ such that $f(b) \geq f(x)$ for all $x \in \operatorname{co}(C_1, C_2)$. Since $b \in \overline{\operatorname{co}(C_1, C_2)}$ there exist $x_n \in C_1$, $x'_n \in C_2$, and $0 \leq \beta_n \leq 1$ such that

$$\lim_{n \to \infty} \beta_n x_n + (1 - \beta_n) x'_n = b.$$

By passing to a subsequece, we may assume that $\{\beta_n\}$ converges to some $\beta$, $0 \leq \beta \leq 1$. It is easy to see that if $\beta \neq 0$ (respectively, $\beta \neq 1$), then

$$\lim_{n \to \infty} f(x_n) = \sup\{f(y) : y \in C_1\}$$
$$(\text{respectively, } \lim_{n \to \infty} f(x'_n) = \sup\{f(y) : y \in C_2\}).$$

But $C_1$ and $C_2$ have the drop property. Hence, if $\beta \neq 0$ (respectively, $\beta \neq 1$), then $\{x_n\}$ (respectively, $\{x'_n\}$) contains a subsequence that converges to some element

$$x \in \{y \in C_1 : f(y) = \sup\{f(z) : z \in C_1\}(= f(b))\}$$
$$(\text{respectively, } x' \in \{y \in C_2 : f(y) = \sup\{f(z) : z \in C_2\}(= f(b))\}).$$

So if $0 < \beta < 1$, then $b = \beta x + (1 - \beta) x' \in \operatorname{co}(C_1, C_2)$. On the other hand, if $\beta = 1$ (respectively, $\beta = 0$), then

$$b = \lim_{n \to \infty} \big(\beta_n x + (1 - \beta_n) x'_n\big) \in \overline{D(x, C_2)}$$
$$(\text{respectively, } b = \lim_{n \to \infty} \big(\beta_n x_n + (1 - \beta_n) x'\big) \in \overline{D(x', C_1)}).$$

By Proposition 5 of [KR1], $D(x, C_2)$ and $D(x', C_1)$ are closed sets. So $b \in \operatorname{co}(C_1, C_2)$; we get a contradiction. □

**Lemma 3.** *Let $C$ be a closed convex set with nonempty interior. If $C$ has property $(\alpha)$, then $C$ has property $(*)$.*

*Proof*. Suppose it is not true. There exist $c \in C$ and $b, x \in X$ such that $b \neq 0$, $\{c + \lambda b : \lambda \geq 0\} \subseteq C$ but $\{x + \lambda b : \lambda \geq 0\} \cap C$ is not a ray. By the simple convexity argument (see [KR1, Proof of Lemma 2]), the line $\{x + \lambda b : \lambda \in \mathbb{R}\}$

is disjoint with $C$. Since $C$ has at least one interior point, by Hahn-Banach Theorem, there is $f \in X^*$ such that

$$\inf\{f(x+\lambda b) : \lambda \in \mathbb{R}\} \geq M = \sup\{f(y) : y \in C\}.$$

This implies $f(b) = 0$, and $S(f, C, M - f(c) + 1)$ contains a ray. We get a contradiction and $C$ must have property $(*)$. □

*Remark* 2. Let $C$ be a closed convex subset of $X$. A ray $r = \{x + \lambda y : \lambda > 0\}$ is said to be an *asymptote* if $r \cap C = \varnothing$, and for any $\epsilon > 0$ there is $N > 0$ such that $\lambda > N$ implies $d(x + \lambda y, C) = \inf\{\|x + \lambda y - c\| : c \in C\} < \epsilon$. Suppose $C$ is a closed convex set with nonempty interior. Then $C$ has property $(*)$ if and only if $C$ does not have any asymptote and the boundary of $C$ does not contain any ray. The proof is left to the reader.

Let $C$ be a closed convex set. $c \in C$ is said to be a *support point* of $C$ if there exists $f \in X^*$, $f \neq 0$, such that $f(c) = \sup\{f(x) : x \in C\}$. A point $c$ in $C$ is said to be a *point of continuity* if for every sequence $\{x_n\}$ in $C$, $\{x_n\}$ converges to $c$ weakly implies $\{x_n\}$ converges to $c$ in norm.

**Theorem 4.** *Let $C$ be a noncompact closed convex subset of a reflexive Banach space. Then the following are equivalent.*

(i) *$C$ has the drop property;*
(ii) $\operatorname{int}(C) \neq \varnothing$ *and $C$ has property $(\alpha)$;*
(iii) $\operatorname{int}(C) \neq \varnothing$, *$C$ has property $(*)$, and every support point $x$ of $C$ is a point of continuity.*

*Proof.* By Theorem A and Lemma 3, we only need to show (iii) implies (ii). First, we claim that for each $f \in F(C)$, $S(f, C, \delta)$ is bounded. Suppose it is not true. There exist $f \in F(C)$ and $\{x_n\} \subseteq C$ such that $\lim_{n\to\infty} \|x_n\| = \infty$ and $\lim_{n\to\infty} f(x_n) = M = \sup\{f(x) : x \in C\}$. Let $y$ be any vector in $X$ such that $\|y\| < 2$ and $f(y) = 1$.

Case 1. There is a subsequence of $\{x_n/\|x_n\|\}$ that conveges weakly to a nonzero vector $b \in X$. Then $r = \{x_1 + \lambda b : \lambda \geq 0\} \subseteq C$ and $f(b) = \lim_{n\to\infty} f(x_n)/\|x_n\| = 0$, but the ray $\{(M+1)y + \lambda b : \lambda > 0\}$ is disjoint with $C$. We get a contradiction.

Case 2. The $\{x_n/\|x_n\|\}$ converges weakly to 0. Without loss of generality, we may assume that 0 is on the boundary of $C$. So $\{x_n/\|x_n\|\}$ converges to 0 in norm. This is impossible, and we prove our claim.

Let $x_n$ be any point in $S(f, C, \frac{1}{n})$. Since $X$ is reflexive, $\{x_n\}$ contains a weakly convergent subsequence $\{x_{n_k}\}$, say it converges to $y \in C$ weakly. Clearly, $f(y) = \sup\{f(x); x \in C\}$. So $y$ is a support point, and $\{x_{n_k}\}$ converges to $y$. This implies $C$ has property $(\alpha)$. □

## 3. Nearly uniform convexity

Recall a closed convex set is said to be (NUC $'$) with a center $a$ if for every $\epsilon > 0$ there exists a $\delta$, $1 > \delta > 0$ such that for every $\epsilon$-separate sequence in $C$, $\overline{\text{co}}(x_n) \cap (a + (1-\delta)(C - a)) \neq \varnothing$. It is easy to see that if $C$ is (NUC) with respect to an $a \in \operatorname{int}(C)$ if and only if $C$ is (NUC $'$) with respect to $a$. In [KR2] D. N. Kutzarova and S. Rolewicz asked whether (NUC) and (NUC $'$) are equivalent. The following theorem shows the answer is affirmative if $C$ has the drop property.

**Theorem 5.** *Let $C$ be a closed convex set with the drop property and $c \in C$. Then $C$ is (NUC) with respect to $c$ if (and only if) $C$ is (NUC$'$) with respect to $c$.*

*Proof.* Since every compact convex set is (NUC), we may assume that the interior of $C$ is nonempty. Let $\{x_n\}$ be an $\epsilon$-seperate sequence in $C$. If $\{x_n\}$ is not bounded, then $\overline{\text{co}}(x_n)$ contains the ray $r = \{x_1 + \lambda b : \lambda \geq 0\}$ for some $b \neq 0$. By Lemma 3, there exists $\beta > 0$ such that $c + 2(x_1 - c + \frac{\beta}{2}b) = c + 2(x_1 - c) + \beta b \in \text{int}(C)$. So $x_1 + \frac{\beta}{2}b \in \overline{\text{co}}(x_n) \cap \text{int}(c + \frac{1}{2}(C - c)) \neq \varnothing$.

If $\{x_n\}$ is bounded, then by passing to a subsequence we may assume $(x_n)$ converges weakly, say it converges to $y \in c + (1-\delta)(C - c)$ weakly. Since $C$ has the drop property, $y$ is an interior point of $C$. This implies $y \in \text{int}(c + (1 - \frac{\delta}{2})(C - c))$ and we prove the theorem. □

*Remark* 3. The proof of the above theorem shows that if $C$ has the drop property, then $C$ is (NUC) with respect to $c$ if and only if it satisfies the following condition:

(o) for any $\epsilon > 0$, there is $\delta$, $0 < \delta < 1$, such that if $x$ is a weak limit of an $\epsilon$-seperate sequence in $C$, then $x \in c + (1 - \delta)(C - c)$.

The following example shows the drop property cannot be removed from the above theorem.

**Example 1.** Let $\{e_n\}$ be the natural basis of $\ell_2$, and let $C$ be the closed convex hull of $\{e_n : n \in \mathbb{N}\}$. Clearly, $0 \in C$. For any $0 < \delta < 1$ and for any $c \in \text{co}\{e_n : n \in \mathbb{N}\}$, $(1-\delta)^{-1}c \notin C$. So $C$ is not (NUC) with respect to 0. We claim that if $x$ is a weak limit of an $\epsilon\sqrt{2}$-sepaeate sequence $\{x_n\} \subseteq C$, then $x \in (1 - \epsilon)C$.

By passing to a subsequence and perturbing $(x_n)$, we may assume that there exists a block squence $\{z_n\}$ such that $x_n = x + z_n$ and $\|z_n\|_2 \geq \epsilon$. But $\|z_n\|_1 \geq \|z_n\|_2$. We have $x \in (1-\epsilon)C$. So $C$ is (NUC$'$) with respect to 0.

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