SOME REMARKS OF DROP PROPERTY

PEI-KEE LIN

(Communicated by William J. Davis)

Abstract. Let $C$ be a proper closed convex set. $C$ is said to have the drop property if for any nonempty closed set $A$ disjoint with $C$, there is $a \in A$ such that $\text{co}(a, C) \cap A = \{a\}$. We show that if $X$ contains a noncompact set with the drop property, then $X$ is reflexive. Moreover, we prove that if $C$ is a noncompact closed convex subset of a reflexive Banach space, then $C$ has the drop property if and only if $C$ satisfies the following conditions: (i) the interior of $C$ is nonempty; (ii) $C$ does not have any asymptote, and the boundary of $C$ does not contain any ray; and (iii) every support point $x$ of $C$ is a point of continuity.

1. Introduction

Let $(X, \| \cdot \|)$ be a real Banach space, and let $C$ be a nonempty proper closed convex subset of $X$. For any $x \notin C$, the drop determined by $x$ is the set $D(x, C) = \text{co}(x, C)$, the convex hull of the set $\{x\} \cup C$. Danes [D] proved that if $C$ is a bounded closed subset of $X$ and $A$ is a closed set at positive distance from $C$, then there exists an $a \in A$ such that $D(a, C) \cap A = \{a\}$. Modifying the assumption, Rolewicz [R1] said a nonempty proper closed set $C$ has the drop property if for every nonempty closed set $A$ disjoint with $C$, there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. The bounded closed convex sets with the drop property are studied in [K1, K2, M, R1, R2]. In [R1] Rolewicz proved that if the closed unit ball of $X$ has the drop property (in this case, we say $X$ has the drop property), then $X$ is reflexive. Kutzarova [K1] extended this result by showing $X$ is reflexive if $X$ contains a noncompact bounded closed convex set (respectively, a noncompact balanced closed convex set) with the drop property. Recently, Kutzarova and Rolewicz [KR1] showed that $X$ is reflexive if $X$ contains a noncompact closed convex symmetric set with the drop property.

For any subset $C$ of $X$, the Kuratowski measure of $C$ is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of $C$ by a finite number of sets of diameter less than $\varepsilon$. It is known that $\alpha(C) = 0$ if and only if $C$ is totally bounded. Let $C$ be a closed convex subset of $X$. We denote the set of all nonzero linear functionals $f \in X^*$, which are bounded above $C$ by $F(C)$.

Received by the editors December 7, 1990.


Key words and phrases. Drop property, reflexive space, compact, nearly uniformly convex.

©1992 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
For any \( f \in F(C) \), and any \( \delta > 0 \), the slice \( S(f, C, \delta) \) is the set
\[
\{ x \in C : f(x) \geq M - \delta \},
\]
where \( M = \sup\{ f(x) : x \in C \} \). A closed convex set \( C \) is said to have property (\( \alpha \)) if
\[
\lim_{\delta \to 0} \alpha(S(f, C, \delta)) = 0
\]
for all \( f \in F(C) \). It is easy to see that a closed convex set \( C \) has property (\( \alpha \)) if and only if for any \( f \in F(C) \) and \( x_n \in S(f, C, \frac{1}{n}) \), \( \{x_n\} \) contains a convergent subsequence. In [KR1] Kutzarova and Rolewicz proved the following

**Theorem A.** Let \( C \) be any closed convex subset of \( X \).

(i) If \( C \) has the drop property, then \( C \) has property (\( \alpha \)).

(ii) If \( C \) is not compact and if \( C \) has the drop property, then \( C \) has nonempty interior.

(iii) Suppose \( X \) is reflexive. If \( C \) has nonempty interior and \( C \) has property (\( \alpha \)), then \( C \) has the drop property.

(iv) Let \( C \) be a closed bounded convex set of a reflexive Banach space. If \( \text{int}(C) \neq \emptyset \) (where \( \text{int}(C) \) is the interior of \( C \)) and every support point of \( C \) is a point of continuity, then \( C \) has drop property.

Using Theorem A, they proved that if \( C_1 \) and \( C_2 \) are any two bounded sets with the drop property, then \( C_1 \cap C_2 \), \( C_1 + C_2 \), and \( \text{co}(C_1, C_2) \) have the drop property. In §2 we show the assumption of boundedness can be removed. Hence, if \( X \) contains a noncompact closed convex set with the drop property, then \( X \) is reflexive. This gives an answer to a question of D. N. Kutzarova and S. Rolewicz [KR1].

Let \( C \) be a closed convex set. \( C \) is said to have property (\( * \)) if \( C \) contains the ray \( \{c + \lambda b : \lambda \geq 0\} \) implies for any \( x \in X \), there is \( \beta > 0 \) such that \( x + (\beta + \lambda)b \in C \) for every \( \lambda \geq 0 \). In §2 we prove that if \( C \) is a noncompact proper closed convex set of a reflexive Banach space, then \( C \) has the drop property if and only if \( \text{int}(C) \neq \emptyset \), \( C \) has property (\( * \)), and every support point of \( C \) is a point of continuity. This gives an extension of Theorem A(iv).

Recall a space \( X \) is said to have the Kadec-Klee property (or property (H)) if on the unit sphere the weakly convergent sequence is convergent in norm (i.e., if \( ||x_n|| = 1 \) and \( x_n \) converges weakly to a unit vector \( x \), then \( x_n \) converges to \( x \) in norm). V. Montesinos [M] proved that \( X \) has the drop property if and only if \( X \) is reflexive and \( X \) has the Kadec-Klee property. Recall that a sequence \( \{x_n\} \) is said to be an \( \epsilon \)-separate sequence for some \( \epsilon > 0 \) if \( \text{sep}(x_n) = \inf\{||x_n - x_m|| : n \neq m\} > \epsilon \). A Banach space \( X \) is said to have the uniform Kadec-Klee property if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x \) is a weak limit of a norm one \( \epsilon \)-separate sequence, then \( ||x|| < 1 - \delta \). A Banach space is said to be nearly uniformly convex (NUC) if for every \( \epsilon > 0 \) there exists a \( \delta \), \( 1 > \delta > 0 \), such that for every sequence \( \{x_n\} \subseteq B \) with \( \text{sep}(x_n) > \epsilon \), we have \( \text{co}(x_n) \cap (1 - \delta)B \neq \emptyset \). It is easy to see that every (NUC) space has the uniform Kadec-Klee property, and every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. Huff [H] proved that \( X \) is (NUC) if and only if \( X \) is reflexive and \( X \) has the uniform Kadec-Klee property. Modifying the theorem, Kutzarova and Rolewicz [KR2] said a closed convex set is (NUC) (respectively (NUC')) with respect to a center \( c \in C \) if
for every $\epsilon > 0$ there exists a $\delta$, $1 > \delta > 0$, such that for every $\epsilon$-separate sequence $\{x_n\} \subseteq C$

$$\text{co}(x_n) \cap (1 - \delta)(C - c) \neq \emptyset$$

(respectively, $\overline{\text{co}}(x_n) \cap (1 - \delta)(C - c) \neq \emptyset$).

It is easy to see that if $C$ is (NUC) with respect to $c \in C$, then $C$ is (NUC') with respect to $c$. Kutzarova and Rolewicz [KR2] proved that if $c$ is an interior point of $C$, then $C$ is (NUC) with respect to $c$ if and only if $C$ is (NUC') with respect to any $c$. They asked whether this is still true if $c$ is a boundary point of $C$. In §3, we show this is true if $C$ has the drop property. We also give an example to show the assumption of the drop property cannot be removed.

2. ON THE DROP PROPERTY

In [KR1] Kutzarova and Rolewicz asked whether $X$ is reflexive if $X$ contains a noncompact closed convex with the drop property. The following theorem shows the answer is affirmative.

**Theorem 1.** Let $C_1$ and $C_2$ be any two closed convex subsets of $X$ with the drop property. If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ has the drop property. Hence, if $X$ contains a noncompact closed convex set with the drop property, then $X$ is reflexive.

**Proof.** Let $A$ be any closed subset of $X$ such that $A \cap (C_1 \cap C_2) = \emptyset$. If $A \cap C_1 = \emptyset$, then there exists $a \in A$ such that $D(a, C_1) \cap A = \{a\}$. This implies $D(a, (C_1 \cap C_2)) \cap A = \{a\}$. So we may assume that $A \cap C_1 \neq \emptyset$. Since $(A \cap C_1) \cap C_2 = \emptyset$ and $C_2$ has the drop property, there is $a \in A \cap C_1$ such that $D(a, C_2) \cap (A \cap C_1) = \{a\}$. So

$$D(a, (C_2 \cap C_1)) \cap A \subseteq (D(a, (C_2 \cap C_1))) \cap C_1 \cap A = \{a\},$$

and $C_1 \cap C_2$ has the drop property.

It is easy to see that if $X$ contains a noncompact closed convex set with the drop property, then $X$ contains a noncompact symmetric closed convex set with the drop property. By [KR1, Proposition 4], $X$ is a reflexive space. □

**Remark 1.** Let $C$ be an unbounded closed convex set of a reflexive space. Kutzarova and Rolewicz proved that if $S(f, C, 1)$ is bounded for some $f \in F(C)$, then $C$ contains a ray $\{c + \beta b : \beta > 0\}$. Moreover, if $c' \in C$, $C$ also contains the ray $\{c' + \beta b : \beta > 0\}$.

Let $C_1$ and $C_2$ be any two bounded closed convex sets with the drop property. In [KR1] Kutzarova and Rolewicz proved that $\lambda C_1 + \mu C_2$ and $\text{co}(C_1, C_2)$ have the drop property. The following theorem shows that the boundedness can be removed.

**Theorem 2.** Let $C_1$ and $C_2$ be any two closed convex sets with the drop property. If $\text{co}(C_1, C_2) \neq X$, then

(i) for any $\lambda, \mu \neq 0$, $\lambda C_1 + \mu C_2$ is closed, and it has the drop property;

(ii) $\text{co}(C_1, C_2)$ is closed, and it has the drop property.

**Proof.** We only prove (ii) and leave the proof of (i) to the reader. If $C_1$ and $C_2$ are compact, then $\text{co}(C_1, C_2)$ is compact. So we may assume that $\text{co}(C_1, C_2)$ has an interior point.
First, we show $\text{co}(C_1, C_2)$ has property $(\alpha)$. If $f \in F(\text{co}(C_1, C_2))$, then $f \in F(C_1) \cap F(C_2)$. Let $x$ (respectively, $x'$) be a point in $C_1$ (respectively, $C_2$) such that

$$f(x) = \sup\{f(y) : y \in C_1\}$$
(respectively, $f(x') = \sup\{f(y) : y \in C_2\}$).

One can easily show that

$$S(f, \text{co}(C_1, C_2), \delta) = \text{co}(x, x') + (S(f, C_1, \delta) - x) + (S(f, C_2, \delta) - x').$$

(Compare with the proof of [KR1, Theorem 9 (iii)].) So

$$\lim_{\delta \to 0} \alpha(S(f, \text{co}(C_1, C_2), \delta) = 0$$

and

$$\text{co}(C_1, C_2) \text{ has property } (\alpha).$$

Suppose that $b \in \text{co}(C_1, C_2) \setminus \text{co}(C_1, C_2) \neq \emptyset$. By Hahn-Banach Theorem, there is a linear functional $f$ such that $f(b) \geq f(x)$ for all $x \in \text{co}(C_1, C_2)$. Since $b \in \text{co}(C_1, C_2)$ there exist $x_n \in C_1$, $x'_n \in C_2$, and $0 \leq \beta_n \leq 1$ such that

$$\lim_{n \to \infty} \beta_n x_n + (1 - \beta_n)x'_n = b.$$ 

By passing to a subsequence, we may assume that $\{\beta_n\}$ converges to some $\beta$, $0 \leq \beta \leq 1$. It is easy to see that if $\beta \neq 0$ (respectively, $\beta \neq 1$), then

$$\lim_{n \to \infty} f(x_n) = \sup\{f(y) : y \in C_1\}$$
(respectively, $\lim_{n \to \infty} f(x'_n) = \sup\{f(y) : y \in C_2\}$).

But $C_1$ and $C_2$ have the drop property. Hence, if $\beta \neq 0$ (respectively, $\beta \neq 1$), then $\{x_n\}$ (respectively, $\{x'_n\}$) contains a subsequence that converges to some element

$$x \in \{y \in C_1 : f(y) = \sup\{f(z) : z \in C_1\} (= f(b))\}$$
(respectively, $x' \in \{y \in C_2 : f(y) = \sup\{f(z) : z \in C_2\} (= f(b))\}$).

So if $0 < \beta < 1$, then $b = \beta x + (1 - \beta)x' \in \text{co}(C_1, C_2)$. On the other hand, if $\beta = 1$ (respectively, $\beta = 0$), then

$$b = \lim_{n \to \infty} (\beta_n x_n + (1 - \beta_n)x'_n) \in D(x, C_2)$$
(respectively, $b = \lim_{n \to \infty} (\beta_n x_n + (1 - \beta_n)x'_n) \in D(x', C_1)$).

By Proposition 5 of [KR1], $D(x, C_2)$ and $D(x', C_1)$ are closed sets. So $b \in \text{co}(C_1, C_2)$; we get a contradiction. \qed

**Lemma 3.** Let $C$ be a closed convex set with nonempty interior. If $C$ has property $(\alpha)$, then $C$ has property $(\ast)$.

**Proof.** Suppose it is not true. There exist $c \in C$ and $b, x \in X$ such that $b \neq 0$, \{c + \lambda b : \lambda \geq 0\} $\subseteq C$ but \{x + \lambda b : \lambda \geq 0\} $\cap C$ is not a ray. By the simple convexity argument (see [KR1, Proof of Lemma 2]), the line \{x + \lambda b : \lambda \in \mathbb{R}\}
is disjoint with \( C \). Since \( C \) has at least one interior point, by Hahn-Banach Theorem, there is \( f \in X^* \) such that
\[
\inf\{f(x + \lambda b) : \lambda \in \mathbb{R}\} \geq M = \sup\{f(y) : y \in C\}.
\]
This implies \( f(b) = 0 \), and \( S(f, C, M - f(c) + 1) \) contains a ray. We get a contradiction and \( C \) must have property (\( * \)). \( \square \)

**Remark 2.** Let \( C \) be a closed convex subset of \( X \). A ray \( r = \{x + \lambda y : \lambda > 0\} \) is said to be an asymptote if \( r \cap C = \emptyset \), and for any \( \epsilon > 0 \) there is \( N > 0 \) such that \( \lambda > N \) implies \( d(x + \lambda y, C) = \inf\{\|x + \lambda y - c\| : c \in C\} < \epsilon \). Suppose \( C \) is a closed convex set with nonempty interior. Then \( C \) has property (\( * \)) if and only if \( C \) does not have any asymptote and the boundary of \( C \) does not contain any ray. The proof is left to the reader.

Let \( C \) be a closed convex set. \( c \in C \) is said to be a support point of \( C \) if there exists \( f \in X^* \), \( f \neq 0 \), such that \( f(c) = \sup\{f(x) : x \in C\} \). A point \( c \) in \( C \) is said to be a point of continuity if for every sequence \( \{x_n\} \) in \( C \), \( \{x_n\} \) converges to \( c \) weakly implies \( \{x_n\} \) converges to \( c \) in norm.

**Theorem 4.** Let \( C \) be a noncompact closed convex subset of a reflexive Banach space. Then the following are equivalent.

(i) \( C \) has the drop property;
(ii) \( \text{int}(C) \neq \emptyset \) and \( C \) has property (\( \alpha \));
(iii) \( \text{int}(C) \neq \emptyset \), \( C \) has property (\( * \)), and every support point \( x \) of \( C \) is a point of continuity.

**Proof.** By Theorem A and Lemma 3, we only need to show (iii) implies (ii). First, we claim that for each \( f \in F(C) \), \( S(f, C, \delta) \) is bounded. Suppose it is not true. There exist \( f \in F(C) \) and \( \{x_n\} \subseteq C \) such that \( \lim_{n \to \infty} \|x_n\| = \infty \) and \( \lim_{n \to \infty} f(x_n) = M = \sup\{f(x) : x \in C\} \). Let \( y \) be any vector in \( X \) such that \( \|y\| < 2 \) and \( f(y) = 1 \).

Case 1. There is a subsequence of \( \{x_n/\|x_n\|\} \) that converges weakly to a nonzero vector \( b \in X \). Then \( r = \{x_1 + \lambda b : \lambda \geq 0\} \subseteq C \) and \( f(b) = \lim_{n \to \infty} f(x_n)/\|x_n\| = 0 \), but the ray \( \{(M + 1)y + \lambda b : \lambda > 0\} \) is disjoint with \( C \). We get a contradiction.

Case 2. The \( \{x_n/\|x_n\|\} \) converges weakly to 0. Without loss of generality, we may assume that 0 is on the boundary of \( C \). So \( \{x_n/\|x_n\|\} \) converges to 0 in norm. This is impossible, and we prove our claim.

Let \( x_n \) be any point in \( S(f, C, \frac{1}{n}) \). Since \( X \) is reflexive, \( \{x_n\} \) contains a weakly convergent subsequence \( \{x_{n_k}\} \), say it converges to \( y \in C \) weakly. Clearly, \( f(y) = \sup\{f(x) : x \in C\} \). So \( y \) is a support point, and \( \{x_{n_k}\} \) converges to \( y \). This implies \( C \) has property (\( \alpha \)). \( \square \)

### 3. Nearly uniform convexity

Recall a closed convex set is said to be (NUC') with a center \( a \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for every \( \epsilon \)-separate sequence in \( C \), \( C \cap (a + (1 - \delta)(C - a)) \neq \emptyset \). It is easy to see that if \( C \) is (NUC) with respect to an \( a \in \text{int}(C) \) if and only if \( C \) is (NUC') with respect to \( a \). In [KR2] D. N. Kutzarova and S. Rolewicz asked whether (NUC) and (NUC') are equivalent. The following theorem shows the answer is affirmative if \( C \) has the drop property.
Theorem 5. Let $C$ be a closed convex set with the drop property and $c \in C$. Then $C$ is (NUC) with respect to $c$ if and only if $C$ is (NUC') with respect to $c$.

Proof. Since every compact convex set is (NUC), we may assume that the interior of $C$ is nonempty. Let $\{x_n\}$ be an $\epsilon$-separate sequence in $C$. If $\{x_n\}$ is not bounded, then $\overline{c_0}(x_n)$ contains the ray $r = \{x_1 + \lambda b : \lambda \geq 0\}$ for some $b \neq 0$. By Lemma 3, there exists $\beta > 0$ such that $c + 2(x_1 - c + \frac{\epsilon}{2} b) = c + 2(x_1 - c) + \beta b \in \text{int}(C)$. So $x_1 + \frac{\epsilon}{2} b \in \overline{c_0}(x_n) \cap \text{int}(c + (1 - \frac{\epsilon}{2})(C - c)) \neq \emptyset$.

If $\{x_n\}$ is bounded, then by passing to a subsequence we may assume $\{x_n\}$ converges weakly, say it converges to $y \in c + (1 - \delta)(C - c)$ weakly. Since $C$ has the drop property, $y$ is an interior point of $C$. This implies $y \in \text{int}(c + (1 - \frac{\epsilon}{2})(C - c))$ and we prove the theorem. \(\Box\)

Remark 3. The proof of the above theorem shows that if $C$ has the drop property, then $C$ is (NUC) with respect to $c$ if and only if it satisfies the following condition:

(o) for any $\epsilon > 0$, there is $\delta$, $0 < \delta < 1$, such that if $x$ is a weak limit of an $\epsilon$-separate sequence in $C$, then $x \in c + (1 - \delta)(C - c)$.

The following example shows the drop property cannot be removed from the above theorem.

Example 1. Let $\{e_n\}$ be the natural basis of $\ell_2$, and let $C$ be the closed convex hull of $\{e_n : n \in \mathbb{N}\}$. Clearly, $0 \in C$. For any $0 < \delta < 1$ and for any $c \in \text{co}\{e_n : n \in \mathbb{N}\}$, $(1 - \delta)^{-1}c \notin C$. So $C$ is not (NUC) with respect to $0$. We claim that if $x$ is a weak limit of an $\epsilon\sqrt{2}$-separate sequence $\{x_n\} \subseteq C$, then $x \in (1 - \epsilon)C$.

By passing to a subsequence and perturbing $(x_n)$, we may assume that there exists a block sequence $\{z_n\}$ such that $x_n = x + z_n$ and $\|z_n\|_2 \geq \epsilon$. But $\|z_n\|_1 \geq \|z_n\|_2$. We have $x \in (1 - \epsilon)C$. So $C$ is (NUC') with respect to 0.

References


