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REDUCTION OF PREQUANTIZED Mp^c STRUCTURES

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INTRODUCTION

The Marsden-Weinstein reduction procedure fashions new symplectic manifolds from Hamiltonian actions on old symplectic manifolds. It is of considerable interest to determine the way in which quantization of the reduced phase spaces relates to that of the original phase spaces. In particular, it is natural to ask for conditions under which prequantization data may be passed through the operation of reduction. Note that reduced phase spaces can fail to be metaplectic, so in general it cannot be expected that traditional prequantization data (Kostant-Souriau line bundle plus metaplectic structure: see [2, 3] for example) will pass to reduced phase spaces; moreover, even when reduced phase spaces admit metaplectic structures, these do not generally arise from metaplectic structures on the original phase spaces in a natural manner. Following Hess, we take the view that prequantization data consists of a prequantized Mp^c structure: see [6] for full details. In this note, we describe when and how such prequantization data may be passed to reduced phase spaces . En route, we encounter a generalized version of the Bohr-Wilson-Sommerfeld rule as an obstruction to this passage. We illustrate our account by reference to coadjoint orbits as reduced phase spaces.

1. THE GENERAL CONSTRUCTION

Let (Z, Ω) be a symplectic manifold equipped with a Hamiltonian action of the Lie group G, and suppose the moment map $J: Z \to \mathscr{G}^*$ to be equivariant with respect to the given left action of G on Z and the coadjoint action of G on $\mathscr{G}^*$. Let $\mu \in \mathscr{G}^*$ be a regular value of J, let $S = J^{-1}(\mu)$ be the preimage of μ in Z under J, and let $H = G_\mu$ be the stabilizer of μ in G. Under a variety of assumptions, the orbit space $H\backslash S =: M$ is a manifold and the projection $S \to M$ a submersion; for example, it is enough to assume that the H-action on S is proper and free. We make such an assumption, in which case M acquires a canonical symplectic form ω. The symplectic manifold (M, ω) is called the Marsden-Weinstein reduction (or reduced phase space) of (Z, Ω)

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at μ. If H^0 is the identity component of H then $H^0\backslash S =: M'$ also carries a canonical symplectic form ω' and the symplectic manifold (M', ω') naturally covers (M, ω). For an account of Marsden-Weinstein reduction, we refer to [1]. For our purposes, we need further information on the symplectic geometry of reduced phase spaces.

Fix $x \in S$ and write $m = [x] \in M$ for the H-orbit of x in S. The tangent space to the G-orbit of x at x is precisely the symplectic orthogonal of T_xS in T_xZ:

$$T_x(G \cdot x) = (T_xS)^{\perp};$$

also,

$$T_x(H \cdot x) = (T_xS) \cap T_x(G \cdot x).$$

Consequently,

$$T_x(H \cdot x) = (T_xS) \cap (T_xS)^{\perp}.$$

Moreover, the realization of M as the orbit space $H\backslash S$ gives us a canonical short exact sequence

$$0 \to T_x(H \cdot x) \to T_xS \to T_mM \to 0.$$

If we define

$$E_x := (T_xS)/[(T_xS) \cap (T_xS)^{\perp}]$$

then it follows that we have a canonical isomorphism from E_x to T_mM; this isomorphism pulls back ω_m to the symplectic form on E_x induced from Ω_x on T_xZ. Suppose also that $y \in S$ with $m = [y]$. In this case, $y = h \cdot x$ for some $h \in H$; differentiating the action of h gives an isomorphism h_* from T_xS to T_yS carrying $(T_xS)^{\perp}$ to $(T_yS)^{\perp}$. As a result, there is induced a symplectic isomorphism $h_\# : E_x \to E_y$. Under a variety of assumptions, the isomorphism $h_\#$ is independent of the choice of h such that $y = h \cdot x$; for example, if the H-action on S is free then h is unique. We make such an assumption, in which case it is now clear that the canonical H-action (by #) on the symplectic vector bundle E over S is such that the quotient $H\backslash E$ is canonically TM (as a symplectic vector bundle) just as the quotient of S by H is M.

At the level of symplectic frame bundles, a similar picture emerges. Model (Z, Ω) on the symplectic vector space (V, Ω_0): thus, the fibre of the symplectic frame bundle $\mathrm{Sp}(Z, \Omega)$ over $z \in Z$ consists of all symplectic linear isomorphisms $V \to T_zZ$. If we model $S \subset Z$ on the subspace $W \subset V$ then the bundle $\mathrm{Sp}(Z, \Omega; S)$ of adapted frames has as fibre over $x \in S$ the set of all $b \in \mathrm{Sp}(Z, \Omega)_x$ such that $b(W) = T_xS$; the structure group of $\mathrm{Sp}(Z, \Omega; S)$ is $\mathrm{Sp}(V; W)$—the subgroup of the full symplectic group $\mathrm{Sp}(V)$ consisting of all g with $g(W) = W$. There is a canonical homomorphism from $\mathrm{Sp}(V; W)$ to $\mathrm{Sp}(W/(W \cap W^{\perp}))$; the bundle associated to $\mathrm{Sp}(Z, \Omega; S)$ via this homomorphism is naturally the symplectic frame bundle $\mathrm{Sp}(E)$ when we model E on the symplectic vector space $W/(W \cap W^{\perp})$. Now the symplectic action of G on (Z, Ω) lifts naturally to an action of G on $\mathrm{Sp}(Z, \Omega)$: explicitly, if $g \in G$ and $b \in \mathrm{Sp}(Z, \Omega)$ then $g \cdot b = g_* \circ b$. The resulting action of H maps $\mathrm{Sp}(Z, \Omega; S)$ to itself and induces an action of H on $\mathrm{Sp}(E)$; the quotient of $\mathrm{Sp}(E)$ by H is canonically $\mathrm{Sp}(M, \omega)$.

We remark that obvious changes take place if we factor by the action of H^0 rather than by the action of H. Thus: the quotient of E by H^0 is TM' and the quotient of $\mathrm{Sp}(E)$ by H^0 is $\mathrm{Sp}(M', \omega')$.

In order to introduce prequantization data, it is convenient to interpose some remarks of a group theoretical nature; full details appear in [6]. Recall that if V is a symplectic vector space with symplectic form Ω_0 then the symplectic group $\mathrm{Sp}(V)$ consists of all the linear automorphisms g of V with $g^*\Omega_0 = \Omega_0$. Fix a positive scalar $h = 2\pi\hbar$ and an irreducible projective unitary representation ρ of V on a Hilbert space $\mathbf{H}$ satisfying

$$\rho(x)\rho(y) = \exp\left\{\frac{1}{2i\hbar}\Omega_0(x, y)\right\}\rho(x+y)$$

whenever $x, y \in V$. As a consequence of the uniqueness theorem due to Stone and von Neumann, if $g \in \mathrm{Sp}(V)$ then $\rho \circ g$ is unitarily equivalent to ρ: there exists a unitary operator U on $\mathbf{H}$ such that

$$v \in V \Rightarrow \rho(gv) = U\rho(v)U^{-1};$$

irreducibility of ρ guarantees uniqueness of U up to scalar multiples. $Mp^c(V)$ denotes the group of all such unitaries U on $\mathbf{H}$ as g ranges over $\mathrm{Sp}(V)$; thus, we have a short exact sequence

$$1 \to U(1) \to Mp^c(V) \xrightarrow{\sigma} \mathrm{Sp}(V) \to 1$$

where σ sends U to g. The group $Mp^c(V)$ has a unique unitary character η such that $\eta(\lambda) = \lambda^2$ when $\lambda \in U(1)$. Although the exact sequence for $Mp^c(V)$ does not split, the derived exact sequence of Lie algebras

$$0 \to u(1) \to mp^c(V) \xrightarrow{\sigma_*} \mathrm{sp}(V) \to 0$$

is canonically split by $\frac{1}{2}\eta_* : mp^c(V) \to u(1)$. We remark that the kernel of η is a connected double cover of $\mathrm{Sp}(V)$ known as the metaplectic group.

We are now in a position to introduce prequantization data for the symplectic manifold (Z, Ω). This data comes in two parts: an Mp^c structure (which makes sense for any symplectic vector bundle) with a prequantum form (which essentially assumes a symplectic manifold). For convenience, we briefly describe these notions here; for full details, see [6].

An Mp^c structure for the symplectic vector bundle B over the manifold X is a principal $Mp^c(V)$ bundle P on X together with a σ-equivariant bundle map from P to the symplectic frame bundle $\mathrm{Sp}(B)$. In fact, every symplectic vector bundle $B \to X$ admits Mp^c structures—there is no obstruction to their existence; further, up to the natural notion of equivalence, the Mp^c structures for $B \to X$ are parametrized by $H^2(X; \mathbf{Z})$. We remark that, in contrast, $B \to X$ admits the more familiar metaplectic structures precisely when the Stiefel-Whitney class $w_2(B)$ vanishes.

A prequantized Mp^c structure (P, γ) for (Z, Ω) is an Mp^c structure P for the symplectic vector bundle TZ provided with a prequantum form γ: an Mp^c-invariant $u(1)$-valued one-form on P such that $\gamma(\tilde{z}) = \frac{1}{2}\eta_* z$ and $d\gamma = (1/i\hbar)\pi^*\Omega$, where $z \in mp^c(V)$ determines the fundamental vector field $\tilde{z}$ on P and where $\pi: P \to Z$ is the bundle projection. We remark that a prequantum form γ on P corresponds naturally with a connexion α^γ of curvature $(2/i\hbar)\Omega$ in the principal $U(1)$ bundle $P(\eta)$ associated to P via the unitary character η. Whereas Mp^c structures themselves always exist, (Z, Ω) admits prequantized Mp^c structures iff the real cohomology class $[\frac{\Omega}{\hbar}] - \frac{1}{2}c(\Omega)^{\mathbf{R}}$ is integral, where $c(\Omega)^{\mathbf{R}}$ is the real first Chern class determined by a unitary reduction of TZ; further, when they exist, the inequivalent prequantized Mp^c structures for (Z, Ω) are parametrized by $H^1(Z; U(1))$.

We can address the problem of passing prequantized Mp^c structures from (Z, Ω) to its reduced phase space once we have dealt with a further property of Mp^c groups. Let $Mp^c\,(V; W)$ be the preimage of $\mathrm{Sp}(V; W)$ in $Mp^c(V)$ and note that $W/(W \cap W^{\perp})$ is a distinguished symplectic subspace of $(W + W^{\perp})/(W \cap W^{\perp})$. In [6] we show that the canonical homomorphism ν from $\mathrm{Sp}(V; W)$ to $\mathrm{Sp}((W + W^{\perp})/(W \cap W^{\perp}))$ lifts naturally to a homomorphism $\hat{\nu}$ from $Mp^c(V; W)$ to $Mp^c((W + W^{\perp})/(W \cap W^{\perp}))$; this clearly induces a homomorphism from $Mp^c(V; W)$ to (a copy of) $Mp^c(W/(W \cap W^{\perp}))$, which we also call $\hat{\nu}$. We remark that these lifted homomorphisms do not generally exist at the level of metaplectic covers.

Now, let (Q, δ) be a prequantized Mp^c structure for the symplectic manifold (Z, Ω). The portion Q^S of Q lying over $\mathrm{Sp}(Z, \Omega; S)$ is a principal $Mp^c(V; W)$ bundle on S; associated to Q^S via the canonical homomorphism $\hat{\nu}$ is an Mp^c structure Q_S for the symplectic vector bundle $E = TS/[(TS) \cap (TS)^{\perp}]$ over S. The restriction δ^S of δ to $Q^S \subset Q$ pushes forward to a form δ_S on Q_S; this is quite easily seen, for example, by regarding the prequantum form δ on Q as a connexion α^{δ} in the associated principal bundle $Q(\eta)$. It is from (Q_S, δ_S) that we intend to construct prequantized Mp^c structures for (M', ω') and (M, ω).

Recall that $\mathrm{Sp}(M', \omega')$ is naturally the quotient of $\mathrm{Sp}(E)$ by the canonical action of H^0, the identity component of H. We would like to produce a prequantized Mp^c structure (P', γ') for (M', ω') as a quotient of (Q_S, δ_S). In order to do this, we view δ_S as a connexion in the principal $U(1)$ bundle $Q_S \to \mathrm{Sp}(E)$ and require that its holonomy be trivial on each orbit of H^0 in $\mathrm{Sp}(E)$. When this condition is satisfied, we introduce an equivalence relation on Q_S by declaring that $x \in Q_S$ and $y \in Q_S$ are equivalent iff x and y are ends of the δ_S-horizontal lift of some curve in an H^0-orbit of $\mathrm{Sp}(E)$. The quotient P' of Q_S under this equivalence relation is an Mp^c structure for the reduced phase space (M', ω') on which δ_S descends to define a prequantum form γ'. We remark that the process of reducing a principal $U(1)$ bundle with connexion under a condition of trivial holonomy is well-known: see [7] for example. Note that, since (Q_S, δ_S) comes from (Q^S, δ^S) by a push-forward operation, we may replace our earlier holonomy condition by the requirement that δ^S have trivial holonomy on each orbit of H^0 in $\mathrm{Sp}(Z, \Omega; S)$.

In terms of our established notation, we have demonstrated the following.

Theorem 1. *Let (Q, δ) be a prequantized Mp^c structure for the phase space (Z, Ω). If the induced connexion δ^S in the principal bundle $Q^S \to \mathrm{Sp}(Z, \Omega; S)$ has trivial holonomy on each H^0-orbit in $\mathrm{Sp}(Z, \Omega; S)$, then the reduced phase space (M', ω') naturally acquires a prequantized Mp^c structure (P', γ').*

The vanishing holonomy condition in this theorem is a generalized version of the (corrected) Bohr-Wilson-Sommerfeld quantization rule. Standard versions of the BWS rule are phrased in terms of polarizations; for example, see [7]. In contrast, our generalized version is polarization-independent: it refers only to prequantization data, in the form of prequantized Mp^c structures. Our BWS rule therefore fits quite naturally into the geometric quantization scheme of [6].

Assume the condition of Theorem 1: that δ^S has trivial holonomy on each H^0-orbit in $\mathrm{Sp}(Z, \Omega; S)$. We claim that the action of H^0 on $\mathrm{Sp}(Z, \Omega; S)$ lifts naturally to Q^S and that the quotient of (Q_S, δ_S) by the induced action of H^0 is canonically (P', γ'). Let $q \in Q^S$ and write b for its image in $\mathrm{Sp}(Z, \Omega; S)$. Let $h \in H^0$ and write $h = \exp\xi_n \cdots \exp\xi_1$ for $\xi_1, \ldots, \xi_n$ in the Lie algebra of H. For $0 \le t \le 1$ and $1 \le j \le n$ we put

$$c_j(t) = \exp\, t\xi_j \cdots \exp\xi_1 \cdot b;$$

the concatenation of the paths $c_1, \ldots, c_n$ is a path c in $\mathrm{Sp}(Z, \Omega; S)$ originating at b. We define $h \cdot q$ to be the terminus of the δ^S-horizontal lift of c originating at q. The assumption that δ^S have trivial holonomy on H^0-orbits in $\mathrm{Sp}(Z, \Omega; S)$ ensures that we have well-defined an action of H^0 on Q^S, independently of the way in which elements of H^0 are expressed as products. The orbits of H^0 in the induced action on Q_S are precisely the points of P'. In this way we realize (P', γ') as the quotient of (Q_S, δ_S) under H^0.

Retaining the assumption that δ^S have trivial holonomy on H^0-orbits in $\mathrm{Sp}(Z, \Omega; S)$, let us assume further that the induced H^0-action on Q_S preserving δ_S extends to an H-action on Q_S preserving δ_S. The orbit space $P := H\backslash Q_S$ is an Mp^c structure for TM on which δ_S descends to define a prequantum form γ by virtue of its invariance. In this way, the reduced phase space (M, ω) is provided with a prequantized Mp^c structure. We state this result as follows.

Theorem 2. *Let (Q, δ) be a prequantized Mp^c structure for (Z, Ω). If δ^S has trivial holonomy on H^0-orbits in $\mathrm{Sp}(Z, \Omega; S)$ and the induced H^0-action on (Q_S, δ_S) extends to an H-action on (Q_S, δ_S) then (M, ω) naturally acquires a prequantized Mp^c structure (P, γ).*

We should point out that our theorems only give sufficient conditions in order that prequantization data pass to reduced phase spaces. We should also point out again that our theorems implicitly assume good behavior of the quotients $H\backslash S$ and $H\backslash E$; for example, we might assume the H-action on S to be proper and free.

2. Coadjoint orbits

In this section, we apply our general constructions to the particular case of coadjoint orbits. Of course, for this we must realize coadjoint orbits as reduced phase spaces.

Let G be a Lie group and let $Z = T^*G$ be its cotangent bundle with the canonical symplectic form $\Omega = d\theta$. The standard left action of G on (Z, Ω) is Hamiltonian, with equivariant moment map $J: Z \to \mathscr{G}^*$ given by

$$J(\alpha_g) = Ad'_g\alpha$$

for $\alpha \in \mathscr{G}^*$ and $g \in G$; here, α_g denotes the value at g of the left-invariant one-form α and Ad' denotes the coadjoint representation of G on $\mathscr{G}^*$. Each $\mu \in \mathscr{G}^*$ is a regular value of J; we have

$$S = J^{-1}(\mu) = \{\alpha_g : Ad'_g\alpha = \mu\}$$

and

$$H = G_\mu = \{h \in G : Ad'_h \mu = \mu\}.$$

In fact, if β denotes the right-invariant one-form on G with value μ_e at the identity e, then $\beta : G \to T^*G$ gives a diffeomorphism of G with $\beta(G) = S$; moreover, for the standard left actions of H, β is equivariant. The action of H on S is certainly proper and free. Further, the reduced phase space $M = H\backslash S$ may be canonically identified with the coadjoint orbit $\mathscr{O} = G \cdot \mu \subset \mathscr{G}^*$; indeed, a canonical identification maps $H \cdot \beta(g) \in M$ to $Ad'_{g^{-1}}\mu \in \mathscr{O}$. Similarly, the reduced phase space $M' = H^0\backslash S$ naturally covers the coadjoint orbit $\mathscr{O}$.

Our aim here is to explicitly construct prequantized Mp^c structures for (M', ω') and (M, ω). In order to do so, we must discuss the symplectic frame bundle $\mathrm{Sp}(Z, \Omega)$ and its subbundle $\mathrm{Sp}(Z, \Omega; S)$. It transpires that both bundles are canonically trivial, though the canonical trivializations are hardly obvious.

A canonical diffeomorphism $\Lambda: G \times \mathscr{G}^* \to Z$ is defined by $\Lambda(g, \alpha) = \alpha_g$ for $g \in G$ and $\alpha \in \mathscr{G}^*$. Composing the derivative Λ_* from $T_gG \times T_\alpha\mathscr{G}^*$ to $T_{\alpha_g}Z$ with the natural isomorphism from $\mathscr{G} \times \mathscr{G}^*$ to $T_gG \times T_\alpha\mathscr{G}^*$ gives a canonical isomorphism

$$a_{\alpha_g} : \mathscr{G} \times \mathscr{G}^* \to T_{\alpha_g}Z.$$

A direct calculation reveals that $a^*_{\alpha_g}\Omega_{\alpha_g}$ is the symplectic form Ω_α on $V = \mathscr{G} \times \mathscr{G}^*$ defined by

$$\Omega_\alpha((\xi, \phi), (\eta, \psi)) = \phi(\eta) - \psi(\xi) - \alpha[\xi, \eta],$$

for $\xi, \eta \in \mathscr{G}$ and $\phi, \psi \in \mathscr{G}^*$. It turns out that the symplectic vector spaces (V, Ω_α) and (V, Ω_0) are canonically isomorphic. Indeed, if for $\xi \in \mathscr{G}$ we define $ad'_\xi : \mathscr{G}^* \to \mathscr{G}^*$ by

$$\alpha \in \mathscr{G}^*, \eta \in \mathscr{G} \Rightarrow (ad'_\xi\alpha)(\eta) = \alpha[\eta, \xi],$$

then the isomorphism $F_{\alpha_g} : V \to V$ given by

$$F_{\alpha_g}(\xi, \phi) = \left(\sqrt{2}Ad_{g^{-1}}\xi,\ \frac{1}{\sqrt{2}}Ad'_{g^{-1}}\phi - \frac{1}{\sqrt{2}}ad'_{Ad_{g^{-1}}\xi}\alpha\right)$$

for $\xi \in \mathscr{G}$ and $\phi \in \mathscr{G}^*$ satisfies

$$F^*_{\alpha_g}\Omega_\alpha = \Omega_0.$$

For $\alpha_g \in Z$, the map $b_{\alpha_g} = a_{\alpha_g} \circ F_{\alpha_g}$ is therefore a canonical symplectic isomorphism from (V, Ω_0) to $(T_{\alpha_g}Z, \Omega_{\alpha_g})$.

It is now clear that, modelling (Z, Ω) on (V, Ω_0), the symplectic frame bundle $\mathrm{Sp}(Z, \Omega)$ is canonically trivial. Indeed, b is a canonical global section. As a consequence, (Z, Ω) has a canonical Mp^c structure: the product $Q = Z \times Mp^c(V)$ with σ-equivariant bundle map $Q \to \mathrm{Sp}(Z, \Omega)$ determined by sending (z, I) to b_z for $z \in Z$. If $\pi: Q \to Z$ is the (first factor) bundle projection and ε is the natural flat connexion in $Q \to Z$, then

$$\delta := \frac{1}{i\hbar}\pi^*\theta + \frac{1}{2}\eta_*\varepsilon$$

defines a canonical prequantum form on Q. Thus, (Z, Ω) is endowed with a distinguished prequantized Mp^c structure (Q, δ).

We remark that the action of G on $\mathrm{Sp}(Z, \Omega)$ is given by the effect

$$g \in G, z \in Z \Rightarrow g \cdot b_z = b_{g \cdot z} \circ A_g$$

on the global section b, where $A_g \in \mathrm{Sp}(V)$ is given by

$$A_g(\xi, \phi) = (Ad_g \xi, Ad'_g \phi)$$

for $\xi \in \mathscr{G}$ and $\phi \in \mathscr{G}^*$.

If we let $W \subset V$ be given by $W = \{(\xi, -ad'_\xi \mu) : \xi \in \mathscr{G}\}$, then $W \cap W^\perp = \mathscr{H} \times 0$, where $\mathscr{H}$ is the Lie algebra of H. We shall naturally identify the symplectic vector spaces $W/(W \cap W^\perp)$ and $\mathscr{G}/\mathscr{H}$; under this identification, $\nu(A_h) \in \mathrm{Sp}(W/(W \cap W^\perp))$ corresponds to $a_h \in \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ given by the familiar formula

$$\xi \in \mathscr{G} \Rightarrow a_h(\xi + \mathscr{H}) = Ad_h \xi + \mathscr{H}$$

when $h \in H$.

As a consequence of our choice of F_{α_g} for $g \in G$, the global section b of $\mathrm{Sp}(Z, \Omega)$ defines over S a global section of the principal $\mathrm{Sp}(V; W)$ bundle $\mathrm{Sp}(Z, \Omega; S)$ and so induces a global section of $\mathrm{Sp}(E) \equiv S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$. In applying our theorems from §1, we may therefore consider the holonomy of δ_S on H^0-orbits in $S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ rather than $\mathrm{Sp}(E)$. Note that orbits of H in $S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ take the form $\{(\beta(hg), a_h r) : h \in H\}$, for fixed $g \in G$ and $r \in \mathrm{Sp}(\mathscr{G}/\mathscr{H})$.

We make the following assertion regarding the generalized Keller-Maslov BWS rule encountered in §1.

Claim. δ_S has trivial holonomy on H^0-orbits in $\mathrm{Sp}(E) = S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ iff there exists a homomorphism $\tau : H^0 \longrightarrow Mp^c(W/(W \cap W^\perp)) = Mp^c(\mathscr{G}/\mathscr{H})$ such that the diagram

$$\begin{array}{ccc} & \xrightarrow{\ \tau\ } & Mp^c(\mathscr{G}/\mathscr{H}) \\ H^0 & & \downarrow \sigma \\ & \xrightarrow[\ a\ ]{} & \mathrm{Sp}(\mathscr{G}/\mathscr{H}) \end{array}$$

commutes and such that

$$(\eta \circ \tau)_* = -\frac{2}{i\hbar}\mu$$

on $\mathscr{H}$.

Proof. ($\Leftarrow$) Let τ exist. A typical loop in an H-orbit in $S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ is of the form $c_t = (p_t, a_{h_t} r)$ where $p_t = \beta(h_t g)$, for $0 \le t \le 1$ and fixed $g \in G$, $r \in \mathrm{Sp}(\mathscr{G}/\mathscr{H})$. We claim that the δ_S-horizontal lifts of c are of the form d, given by $0 \le t \le 1 \Rightarrow d_t = (p_t, \tau(h_t)\hat{r})$ for any $\hat{r} \in Mp^c(\mathscr{G}/\mathscr{H})$ with $\sigma(\hat{r}) = r$. Once this is verified, the δ_S-horizontal lifts of the loop c are again loops: ($\Leftarrow$) follows. For the verification, d plainly lifts c since τ lifts a; thus we need only check δ_S-horizontality, and in doing so we may assume $r = I$ and $\hat{r} = I$ for convenience, since horizontality is right invariant. We shall use dots to denote tangent vectors to curves: thus, let $(d/dt)h_t = \dot{h}_t = (z_t)_{h_t}$ for $z_t \in \mathscr{H}$. Since $\beta^*\theta = \beta$ is right invariant and $\beta_h = \mu_h$ for $h \in H$, it follows that

$$\theta_{p_t}(\dot{p}_t) = \beta_{h_t}(\dot{h}_t) = \mu(z_t);$$

also

$$\frac{d}{dt}(\tau(h_t)) = (\tau_* z_t)_{\tau(h_t)}.$$

Thus

$$\delta(\dot{d}_t) = \frac{1}{i\hbar}\theta_{p_t}(\dot{p}_t) + \frac{1}{2}\eta_*(\tau_* z_t) = 0$$

by virtue of the identity $(\eta \circ \tau)_* = -(2/i\hbar)\mu$. This completes the proof of $(\Leftarrow)$.

$(\Rightarrow)$ Let δ_S have trivial holonomy on H^0-orbits in $S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$. Let $h \in H^0$ be joined to the identity e by a path $h_t (0 \le t \le 1)$ with $h_0 = e$ and $h_1 = h$. For any $g \in G$ and $r \in \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ we obtain a path

$$0 \le t \le 1 \Rightarrow c_t = (\beta(h_t g), a_{h_t} r)$$

in $S \times \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ which we then lift δ_S-horizontally to a path

$$0 \le t \le 1 \Rightarrow d_t = (\beta(h_t g), u_t \hat{r})$$

in $S \times Mp^c(\mathscr{G}/\mathscr{H})$ with $u_0 = I$, for any choice of $\hat{r}$ over r. Vanishing holonomy and right invariance ensure that $\tau(h) := u_1$ is a good definition and yields a homomorphism $\tau\colon H^0 \to Mp^c(\mathscr{G}/\mathscr{H})$ as claimed. □

The following is now a direct consequence of Theorem 1 and the above claim.

Theorem 1′. *If the homomorphism $a : H^0 \to \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ lifts to a homomorphism $\tau : H^0 \to Mp^c(\mathscr{G}/\mathscr{H})$ satisfying $(\eta \circ \tau)_* = -(2/i\hbar)\mu|\mathscr{H}$, then the canonical prequantized Mp^c structure (Q, δ) for (Z, Ω) naturally confers a prequantized Mp^c structure (P', γ') on the covering (M', ω') of $\mathscr{O}$.*

Of course, in this context the lifted action of H^0 on $Q_S = S \times Mp^c(\mathscr{G}/\mathscr{H})$ is given by

$$h \cdot (x, u) = (h \cdot x, \tau(h)u)$$

for $h \in H^0$, $x \in S$, and $u \in Mp^c(\mathscr{G}/\mathscr{H})$. This makes plain the following application of Theorem 2.

Theorem 2′. *If $a\colon H \to \mathrm{Sp}(\mathscr{G}/\mathscr{H})$ lifts to a homomorphism $\tau\colon H \to Mp^c(\mathscr{G}/\mathscr{H})$ with $(\eta\circ\tau)_* = -(2/i\hbar)\mu$, then the canonical prequantized Mp^c structure (Q, δ) for (Z, Ω) naturally confers a prequantized Mp^c structure (P, γ) on the coadjoint orbit $\mathscr{O} = G \cdot \mu$.*

Recalling that $\sigma_*\colon mp^c(\mathscr{G}/\mathscr{H}) \to sp(\mathscr{G}/\mathscr{H})$ is canonically split by $\frac{1}{2}\eta_*\colon mp^c(\mathscr{G}/\mathscr{H}) \to u(1)$, we may reformulate the hypothesis on the existence of τ in this theorem as follows: the Lie algebra homomorphism

$$\left(-\frac{1}{i\hbar}\mu\right) \oplus (a_*)\colon \mathscr{H} \longrightarrow mp^c(\mathscr{G}/\mathscr{H})$$

should exponentiate to a Lie group homomorphism $\tau\colon H \to Mp^c(\mathscr{G}/\mathscr{H})$. In this form, the hypothesis can be compared with the familiar condition for $\mathscr{O}$ to inherit a (Kostant-Souriau) prequantum line bundle: namely [5] that $-(1/i\hbar)\mu|\mathscr{H}$ should exponentiate to a unitary character $\chi\colon H \to U(1)$.

References

1. R. Abraham and J. E. Marsden, *Foundations of mechanics*, 2nd ed., Benjamin/ Cummings, Reading, MA, 1978.
2. R. J. Blattner, *Quantization and representation theory*, Proc. Sympos. Pure Math., vol. 26, Amer. Math. Soc., Providence, RI, 1974, pp. 147–165.
3. V. Guillemin and S. Sternberg, *Geometric asymptotics*, Math. Surveys, vol. 14, Amer. Math. Soc., Providence, RI, 1977.

4. H. Hess, *On a geometric quantization scheme generalizing those of Kostant-Souriau and Czy ż*, Lecture Notes in Phys., no. 139, Springer, Berlin and New York, 1981, pp. 1–36.

5. B. Kostant, *Quantization and unitary representations. Part* I: *prequantization*, Lecture Notes in Math., no. 170, Springer, Berlin and New York, 1970, pp. 87–208.

6. P. L. Robinson and J. H. Rawnsley, *The metaplectic representation, Mp^c structures and geometric quantization*, Mem. Amer. Math. Soc., no. 410, Providence, RI, 1989.

7. N. Woodhouse, *Geometric quantization*, Oxford Univ. Press, Oxford, 1980.

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