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EVERY SUPERATOMIC SUBALGEBRA OF AN INTERVAL ALGEBRA IS EMBEDDABLE IN AN ORDINAL ALGEBRA

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ABSTRACT. Let us recall that a Boolean algebra is *superatomic* if every subalgebra is atomic. So by the definition, every subalgebra of a superatomic algebra is superatomic. An obvious example of a superatomic algebra is the interval algebra generated by a well-ordered chain. In this work, we show that every superatomic subalgebra of an interval algebra is embeddable in an ordinal algebra, that is by definition, an interval algebra generated by a well-ordered chain. As a corollary, if B is an infinite superatomic subalgebra of an interval algebra, then B and the set $\mathrm{At}(B)$ of atoms of B have the same cardinality.

1. SURVEY OF THE RESULTS

In a Boolean algebra B we denote by 0_B and 1_B, respectively the smallest and the largest elements of B. For x and y in B, we denote by $x \cup y$ and $x \cap y$, respectively, the supremum and the infimum of x and y in B, by $-x$ the complement of x in B, and $x - y = x \cap (-y)$. So $-x = 1_B - x - x$. Moreover, $x \subset y$ means $x \subseteq y$ and $x \neq y$.

For $0_B \neq a \in B$, we denote by $B \restriction a$ the Boolean algebra induced by B on the set $\{t \in B | t \subseteq a\}$. So $1_{B \restriction a} = a$ and the complement of t in $B \restriction a$ is $a - t$.

For a Boolean algebra B and a subset D of B, we denote by $\mathrm{At}(B)$ the set of atoms B, and by $\mathrm{cl}_B(D)$ the subalgebra of B generated by D.

For example, if I is an ideal of B, then $\mathrm{cl}_B(I) = I \cup -I$, where $-I = \{-x \in B | x \in I\}$.

Definition. A Boolean algebra B is said to be superatomic if every quotient of B is atomic.

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Day [3] (see also Koppelberg [5] and Roitman [7]) has shown the following result:

Proposition. *Let B be a Boolean algebra. The following properties are equivalent:*

(i) *B is superatomic;*
(ii) *every subalgebra of B is atomic; and*
(iii) *there is no embedding from the atomless countable algebra into B.* □

If E is a set, then $\wp(E)$, the power set of E, is regarded as a Boolean algebra.

Let $(C, \leq)$ be a partial ordered set. We say that $(C, \leq)$ is a *chain* if every pair of members of C are comparable, and $(C, \leq)$ is well ordered if $(C, \leq)$ has no strictly decreasing sequence. Let $(C, \leq)$ be a chain with a first element denoted by 0_C (if $(C, \leq)$ has no first element, then we must add one). Let $C^+ \stackrel{\text{def}}{=} C \cup \{\infty_C\}$ be the chain, obtained by adding a greatest element ∞_C. We denote by $B(C)$ the subalgebra of $\wp(C)$ generated by the set of $[a, b)$ for $a \in C$ and $b \in C^+$, i.e. $\mathrm{cl}_{\wp(C)}(\{[a, b) | a \in C \text{ and } b \in C^+\})$. $B(C)$ is called the *interval algebra* on C (see Koppelberg [5]).

Theorem 1. *Let B be a superatomic Boolean algebra. If B is embeddable in an interval algebra, then B is embeddable in an interval algebra generated by a well-ordered chain C. More precisely, B is isomorphic to a subalgebra B' of $B(C)$ and $\mathrm{At}(B') = \mathrm{At}(B(C))$.*

Obviously Theorem 1 holds if B is countable, since B is isomorphic to an interval algebra (see Mayer and Pierce [6], Koppelberg [5]).

Example and comment. Let B be a superatomic subalgebra of an interval algebra $B(C)$. The question is as follows: *Is there a superatomic interval algebra A such that $B \subseteq A \subseteq B(C)$?* The answer is negative. Indeed, let $C = 2 \cdot \lambda$, where λ denotes the chain of real numbers, and thus C is the chain obtained from the chain λ by replacing each real number by the 2-elements chain. Let $B \stackrel{\text{def}}{=} \mathrm{cl}_{B(C)}(\mathrm{At}(B(C)))$. Let $A \stackrel{\text{def}}{=} B(D)$ be a superatomic interval algebra, generated by a chain D, containing B. First, because $\mathrm{At}(B)$ is uncountable, the scattered chain D is uncountable. By a theorem of Hausdorff (see Rosenstein [8, Theorem 5.28]), D contains a copy of the chain ω_1 or ω_1^*. For a contradiction, let us suppose that $B(D) \subseteq B(2 \cdot \lambda)$. Because $B(D)$ satisfies: there is an uncountable family of pairwise disjoint elements such that each element contains infinitely many atoms, the algebra $B(2 \cdot \lambda)$ has the same property. Now, we obtain a contradiction with the fact that in λ, every family of pairwise disjoint nontrivial intervals is at most countable.

For an infinite well-ordered chain C, the sets $\mathrm{At}(B(C))$ and $B(C)$ have the same cardinality; hence Theorem 1 implies the following result, proved by M. Rubin and S. Shelah [1988]:

Theorem 2. *Let B be an infinite superatomic Boolean algebra, embeddable in an interval algebra. Then $\mathrm{At}(B)$ and B have the same cardinality.*

This result completes the different characterizations of superatomic subalgebras of an interval algebra, developed by R. Bonnet, M. Rubin, and H. Si-Kaddour [1988]. Let us give an application of the above theorem. A Boolean algebra B is said to be the *thin-tall* if B is uncountable, and for each ordinal α, the set $\mathrm{At}(D_\alpha(B))$ is countable (the algebra $D_\alpha(B)$ is defined in 2.3). In particular $\mathrm{At}(B)$ is countable. From Theorem 2, for example, it follows that a thin-tall Boolean algebra is not embeddable in an interval algebra (for an application, see R. Bonnet and S. Shelah [2]).

2. Notation and definition

2.1. **Chains.** Let $(C, \leq)$ be a chain, $u \in C$, and D be a subset of C.

$(C, \leq)$ is a *complete chain* if every subset of C has a supremum and an infimum.

$(C, \leq)$ is a *relatively complete chain* if every bounded nonempty subset A of C (i.e. there are b and c in C such that $b \leq a \leq c$ for every $a \in A$) has a supremum and an infimum in C.

Assume that C is complete. We denote by $c(D, C)$, or more simply $c(D)$, the closure of D in C by supremum and infimum. For the following result, see e.g. Rosenstein [8]:

Every chain $(C, \leq)$ is embeddable in a complete chain $(C^d, \leq)$; namely, its *Dedekind completion*, (completion by cuts), which satisfies: for every $c \in C^d$, if c is not the first element of C^d, then $c = \sup\{p \in C | p \leq c\}$, and if c is not the last element of C^d, then $c = \inf\{p \in C | p \geq c\}$.

u is a *predecessor in* C if there is an (unique) element $u^+ \in C$ such that $u^+ > u$ and $[u, u^+) = \{u\}$. We denote by $\mathrm{Pred}(C)$ the set of predecessors of C.

$(C, \leq)$ is *totally disconnected* if for every $v < w$ in C, we have $[v, w) \cap \mathrm{Pred}(C) \neq \varnothing$. Consequently $B(C)$ is atomic if and only if C is totally disconnected. The word "totally disconnected" comes from the fact that a chain C is totally disconnected if and only if C, endowed with the interval topology, is a totally disconnected space.

Let $\cong$ be an equivalence relation on C. For $a \in C$, we denote by $a/\cong$ its equivalence class, i.e. $a/\cong \stackrel{\text{def}}{=} \{a' \in C | a' \cong a\}$. We suppose that each equivalence class is an interval of C. For $\tilde{a}', \tilde{a}'' \in C/\cong$ we set $\tilde{a}' < \tilde{a}''$ if for every $a' \in \tilde{a}'$ and $a'' \in \tilde{a}''$, we have $a' < a''$. *If each equivalence class is an interval of* C, *then* $(C/\cong, \leq)$ *is a chain. Moreover, if* $(C, \leq)$ *is a complete chain, then* $(C/\cong, \leq)$ *is too.*

2.2. **Boolean algebras.** An element a of an interval algebra $B(C)$, different from 0_B, has a unique decomposition (called the *canonical decomposition*), of the form: $a = \bigcup\{[a_{2i}, a_{2i+1}) | i < n\}$ where $0 < n < \omega$, $0_C \leq a_0 < a_1 < a_2 < \cdots < a_{2n-1} \leq \infty_C$, and $a_k \in C^+ \stackrel{\text{def}}{=} C \cup \{\infty_C\}$, $(k = 0, 1, \ldots, 2n-1)$. For such an element a, we set $\sigma(a) = \{a_k | k < 2n\} \subseteq C \cup \{\infty_C\}$. The integer n is called the *length of* a, and is denoted by $l(a)$.

Every finite product of interval algebras is isomorphic to an interval algebra, and thus *if* B *is a subalgebra of a finite product of interval algebras, then* B *is embeddable in an interval algebra.*

More precisely, let us recall the following fact concerning chains and Boolean algebras (see [5, Proposition 15.11]). Let C_1 and C_2 be chains with first element 0_{C_1} and 0_{C_2} respectively. Let $C = C_1 + C_2$ be the chain, lexicographic sum of C_1 and C_2 (so $c_1 < c_2$ for $c_1 \in C_1$ and $c_2 \in C_2$). Note that C has a first element, namely 0_{C_1}. A canonical isomorphism f from $B(C)$ onto $B(C_1) \times B(C_2)$ is obtained by letting: $f(c) = \langle c \cap C_1, c \cap C_2 \rangle$. Let us remark that we identified ∞_{C_1} with 0_{C_2}. $B(C_1)$, $B(C_2)$ are factors of $B(C)$; and by identification, $B(C) \restriction C_1 = B(C_1)$ and $B(C) \restriction C_2 = B(C_2)$.

Let D be a subset of C, containing 0_C. Hence D is a chain with a first element. We denote by $B_C(D)$ the subalgebra of $B(C)$ consisting of those elements a such that $\sigma(a) \subseteq D \cup \{\infty_C\}$. Let us remark that the Boolean algebras $B_C(D)$ and $B(D)$ are isomorphic.

2.3. Let B be a Boolean algebra. By induction, we define a sequence $(I_\alpha(B), D_\alpha(B), \pi_\alpha^B)$, with the conditions $D_\alpha(B) = B/I_\alpha(B)$ and π_α^B is the canonical homomorphism from B onto $D_\alpha(B)$ (the algebra $D_\alpha(B)$ is called the *α-th Cantor Bendixon derivative of* B). Let $I_0(B) = \{0\}$, and thus $D_0(B) = B$. Suppose that $I_\alpha(B)$ has been defined. Let $J_\alpha(B)$ be the ideal of $D_\alpha(B)$, generated by $\mathrm{At}(D_\alpha(B))$. Then $I_{\alpha+1}(B) \overset{\text{def}}{=} (\pi_\alpha^B)^{-1}(J_\alpha(B))$. Suppose that δ is a limit, and $I_\alpha(B)$ has been defined for every $\alpha < \delta$, then $I_\delta(B) = \bigcup_{\alpha<\delta} I_\alpha(B)$.

The following additional equivalences are well known and their proofs are straightforward (see Koppelberg [5]).

Proposition. *Let B be a Boolean algebra. The following properties are equivalent:*

(i) *B is superatomic, and*
(ii) *there is an ordinal γ such that $1_B \in I_\gamma(B)$.* □

Clearly the first ordinal γ for which $1_B \in I_\gamma(B)$ is a successor ordinal, say $\alpha+1$, and α is α is denoted by $\mathrm{rk}(B)$. Hence $1_B \in I_{\mathrm{rk}(B)+1}(B) - I_{\mathrm{rk}(B)}(B)$ and $D_{\mathrm{rk}(B)}(B)$ is a nontrivial finite algebra isomorphic to $\wp(n)$ ($n > 0$ integer), and if $n = 1$, then $I_{\mathrm{rk}(B)}(B)$ is a maximal ideal of B. Let $I(B)$ and $D(B)$ denote $I_{\mathrm{rk}(B)}(B)$ and $B/I(B)$ respectively.

For $b \in B$, $b \neq 0_B$ let $\mathrm{rk}_B(b)$ be the first ordinal α such that $b \notin I_\alpha(B)$. Hence $b \in I_{\mathrm{rk}_B(b)+1}(B) - I_{\mathrm{rk}_B(b)}(B)$. For instance $\mathrm{rk}_B(b) = 0$ for $b \in \mathrm{At}(B)$, and $\mathrm{rk}_B(1_B) = \mathrm{rk}(B)$.

Notation 2.4. Let B be a subalgebra of an algebra A, and $c \in A$. We denote by $B \restriction c$ the set of $b \cap c$ for $b \in B$. We regard $B \restriction c$ as a subalgebra of the factor $A \restriction c$ of A, and thus as a Boolean algebra. Remark that if $c \in B$, then $B \restriction c$ is a factor of B.

By the definition, $B \restriction c$ is an homomorphic image of B. From the fact that, for every superatomic Boolean algebra A and every ideal I of A, we have $\mathrm{rk}(A/I) \leq \mathrm{rk}(A)$, it follows that:

Lemma 2.5. *Let B be a subalgebra of a superatomic Boolean algebra A, and $c \in A$. Then* $\mathrm{rk}(B \restriction c) \leq \mathrm{rk}(B) = \mathrm{rk}_B(1_B)$. □

The following result is due to M. Rubin and S. Shelah [9], and is one of the ingredients of the original proof of Theorem 2.

Proposition 2.6 (Rubin and Shelah). *Let B be an atomic subalgebra of an interval algebra. Then there are a totally disconnected complete chain C and an embedding ϕ from B into $B(C)$ such that $B(C)$ is an atomic algebra and* $\mathrm{At}(\phi[B]) = \mathrm{At}(B(C))$.

Note that the property of C implies that $c(\bigcup \mathrm{At}(B(C)), C) = C$. The proof of Proposition 2.6 needs some preliminary results. Let C be a chain such that $B \subseteq B(C)$.

Claim 2.7. We can suppose that C satisfies (1): C is a complete chain, and (2): every atom of B is a finite subset of C.

Proof. We can suppose that C is a complete chain (consider its Dedekind completion). Let $\underline{C} = \bigcup\{\sigma(a) | a \in B\}$ be the set of endpoints of elements of B. We set $\underline{C}^c = c(\underline{C}, C)$. The function ϕ from B into the subalgebra $B_{\underline{C}^c}(\underline{C})$ of $B(\underline{C}^c)$ defined by $\phi(b) = b \cap \underline{C}$ is trivially a one-to-one homomorphism, and $\underline{C}^c$ is as required (note that by the construction, (2) is satisfied). □

Claim 2.8. We can suppose that C satisfies (1) and (3): every atom of B is a singleton of C.

Proof. For every $a \in \mathrm{At}(B)$, we have $a = \bigcup_{i<l(a)}[a_{2i}, a_{2i+1}) = \bigcup_{i<l(a)}\{a_{2i}\}$ with $a_k \in C^+ = C \cup \{\infty_C\}$, and a_{2i}, a_{2i+1} consecutive in C. Let $\sim$ be the equivalence on C defined by $a_{2i} \sim a_{2i+1}$ for $0 < i < l(a)$ and $a \in \mathrm{At}(B)$. Let $\underline{C} = C/\sim$. Hence $\underline{C}$, with the induced linear order by C, is a complete chain. Let φ be the function from $B(C)$ into $B(\underline{C})$ defined as follows: if $b = \bigcup_{i<l(b)}[b_{2i}, b_{2i+1})$, then $\varphi(b) = \bigcup_{i<l(b)}[b_{2i}/\sim, b_{2i+1}/\sim)$. Obviously φ is a homomorphism from $B(C)$ onto $B(\underline{C})$. It suffices to show that $\varphi(a) = [a_0/\sim, a_1/\sim) = \{a_0/\sim\} \neq 0$ for $a \in \mathrm{At}(B)$, and φ restricted to B is one-to-one. But this is trivial. □

Proof of Proposition 2.6. To prove the proposition, there is no loss in assuming that B and C satisfy the assumptions (1) and (3) of Claim 2.8. Let $\equiv$ be the equivalence on C defined by $x \equiv y$ if $x = y$, or if $x \leq y$ and $[x, y)$ does not contain an atom of B, or if $y \leq x$ and $[y, x)$ does not contain an atom of B. Then the quotient chain $\underline{C} \stackrel{\text{def}}{=} C/\equiv$ is complete and totally disconnected. Let ρ be the canonical increasing function from C onto $\underline{C}$. Note that if $x < y$ in C are such that $\rho(x) < \rho(y)$ in $\underline{C}$, then there is $a \in \mathrm{At}(B)$ such that $a \subseteq [x, y)$. This shows that the function ϕ from B into $B(\underline{C})$ defined by $\phi(a) = \bigcup\{[\rho(a_{2i}), \rho(a_{2i+1})) | i < l(a)\}$ for $a = \bigcup\{[a_{2i}, a_{2i+1}) | i < l(a)\}$ in $B \subseteq B(C)$, is as required, and satisfies $\mathrm{At}(\phi[B]) = \mathrm{At}(B(\underline{C}))$. That finishes the proof of Proposition 2.6. □

3. Proof of Theorem 1

3.1. To prove Theorem 1, there is no loss in assuming that (1): B satisfies both the premises and the conclusions of Proposition 2.6, and (2): $I(B) = I_{\mathrm{rk}(B)}(B)$ is a maximal ideal of B. We denote by $\cong$ the relation on C defined by $x \cong y$ if $x \leq y$ and there is $b \in B$, with $\mathrm{rk}_B(b) < \mathrm{rk}_B(1_B)$, containing $[x, y)$, or $y \leq x$ and there is $b \in B$, with $\mathrm{rk}_B(b) < \mathrm{rk}_B(1_B)$, containing $[y, x)$. We show:

Lemma 3.2. *Each equivalence class is an interval of* C. *Let* $b \in B$ *be such that* $\mathrm{rk}_B(b) < \mathrm{rk}_B(1_B)$. *Then there is a finite subset* $a_0/\cong, a_1/\cong, \ldots, a_{n-1}/\cong$ *of equivalence classes such that* $b \subseteq \bigcup\{a_k/\cong | k < n\}$.

Proof. The first part of the claim is trivial. Let us show the second one. Let

$b = \bigcup\{[b_{2i}, b_{2i+1})|i < n\}$. It suffices to show that $b_{2i} \cong b_{2i+1}$ for $i < n$. But this is a trivial consequence of the definition of $\cong$. □

The following two claims are obvious.

Claim 3.3. If $a \in \mathrm{At}(B)$, then a is contained in an equivalence class. □

Claim 3.4. Let $\tilde{a}$ be an equivalence class. If $\tilde{a}$ has a last element v, then $v \notin \mathrm{Pred}(C)$. □

Definition 3.5. Let C be a complete totally disconnected chain, B a superatomic subalgebra of $B(C)$, λ an ordinal, and ψ a function from B into $B(\lambda)$. We say that (B, C, λ, ψ) is a good system if $\mathrm{At}(B) = \mathrm{At}(B(C))$, ψ is a one-to-one homomorphism from B into the interval algebra $B(\lambda)$, and the restriction $\psi \restriction \mathrm{At}(B)$ of ψ on $\mathrm{At}(B)$ is a one-to-one function from $\mathrm{At}(B)$ onto $\mathrm{At}(B(\lambda))$. We say that there is a good system for (B, C) if there are λ and ψ such that (B, C, λ, ψ) is a good system.

Note that $\mathrm{At}(\psi[B]) = \psi[\mathrm{At}(B)]$. Equivalently a good system is the (B, C, λ, ψ_0), where $\mathrm{At}(B) = \mathrm{At}(B(C))$, and ψ_0 is a one-to-one function from $\mathrm{Pred}(C)$ into the chain λ such that the function $\underline{\psi}_0$ from $\mathrm{At}(B(C))$ into $\mathrm{At}(B(\lambda))$ defined by $\underline{\psi}_0([u, u^+)) = [\psi_0(u), \psi_0(u)+1)$ for $u \in \mathrm{Pred}(C)$ can be extended in an embedding ψ from B into $B(\lambda)$.

We prove by induction on α, that the following statement $\mathrm{Th}(\alpha)$ holds: *for every chain C and for every superatomic subalgebra B of $B(C)$, such that* $\mathrm{rk}(B) \le \alpha$ *and* $\mathrm{At}(B) = \mathrm{At}(B(C))$, *there is a good system for* (B, C).

$\mathrm{Th}(0)$ and $\mathrm{Th}(1)$ hold. Indeed B is isomorphic to the Boolean algebra $F_c(X)$ of finite or cofinite subsets of a set X, where $X = \mathrm{At}(B(C))$ (since $I(B) = I_{\mathrm{rk}(B)}(B)$ is a maximal ideal of B). Consider λ be the (initial) ordinal corresponding to the cardinality of the set $\mathrm{At}(B(C))$. In what follows, we suppose that $\mathrm{rk}(B) \ge 2$.

Claim 3.6. Let $\tilde{a}/cong$ be an equivalence class, and

$$\tilde{a} \overset{\mathrm{def}}{=} ((a/\cong) \cup \{\inf(a/\cong)\}) - \{\max(a/\cong)\} = [\inf(a/\cong), \sup(a/\cong)).$$

There is a good system for for $(B \restriction \tilde{a}, \tilde{a})$.

Proof. By induction. If $\tilde{a} = \{a\}$, then it is trivial. Assume $|\tilde{a}| \ge 2$. Let $c \in \mathrm{Pred}(\tilde{a})$, $\tilde{a}^+ = \{x \in \tilde{a}|x > c\}$, and $\tilde{a}^- = \{x \in \tilde{a}|x \le c\}$. Note that $\tilde{a}$ is the lexicographic sum $\tilde{a}^- + \tilde{a}^+$, and $\tilde{a}^+$ has a first element (the successor of c). If $\tilde{a}^-$ has no first element, then we must add one, namely $\inf(a/\cong)$. Suppose that $(B \restriction \tilde{a}^+, \tilde{a}^+, \lambda^+, \psi^+)$ and $(B \restriction \tilde{a}^-, \tilde{a}^-, \lambda^-, \psi^-)$ are good. Let $(B \restriction \tilde{a}, \tilde{a}, \lambda^- + \lambda^+, \psi)$, where ψ is defined in the following way: for $b \in B \restriction \tilde{a}$, we have $b = \langle b^-, b^+\rangle \in B(\tilde{a}^-) \times B(\tilde{a}^+)$ and we set $\psi(b) = \langle \psi^-(b^-), \psi^+(b^+)\rangle \in B(\lambda^-) \times B(\lambda^+)$ (that is identified with $B(\lambda^- + \lambda^+)$). Trivially, $(B \restriction \tilde{a}, \tilde{a}, \lambda^- + \lambda^+, \psi)$ is as required. So, it suffices to prove that Claim 3.6, whenever $\tilde{a}^+ = \tilde{a}$ or $\tilde{a}^- = \tilde{a}$. We prove the case $\tilde{a}^+ = \tilde{a}$. The case $\tilde{a}^- = \tilde{a}$ is similar: but note that if $a/\cong$ has no first element, then $\tilde{a} = (a/\cong) \cup \{\inf(a/\cong)\}$, and the algebras

$$B \restriction (a/\cong) \quad \text{and} \quad B \restriction ((a/\cong) \cup \{\inf(a/\cong)\})$$

are isomorphic. $\tilde{a}^+ = \tilde{a}$ satisfies: $\tilde{a}$ has a first element, denoted by e, and for every element x of $\tilde{a}$, we have $x \cong e$.

Case 1. $a/\cong$ has a last element e^+. Hence $e^+ \cong e$ and $\tilde{a} \stackrel{\text{def}}{=} (a/\cong) - \{e^+\} = [e, e+)$ (that is the case of the example which follows from Theorem 1). Let $b \in B$ be such that $\tilde{a} \subseteq b$, and $\mathrm{rk}_{B \restriction b}(1_{B \restriction b}) = \mathrm{rk}_B(b) < \mathrm{rk}_B(1_B)$. Let $B \restriction \tilde{a} \stackrel{\text{def}}{=} \{c \cap \tilde{a} | c \in B\}$. Note that $B \restriction \tilde{a} = (B \restriction b) \restriction \tilde{a}$. We regard $B \restriction \tilde{a}$ as a Boolean algebra. By the definition, $B \restriction \tilde{a}$ is a homomorphic image of $B \restriction b$. From the fact that for every superatomic Boolean algebra A and every ideal I of A, we have $\mathrm{rk}(A/I) \leq \mathrm{rk}(A)$, it follows that $\mathrm{rk}(B \restriction \tilde{a}) \leq \mathrm{rk}(B \restriction b) = \mathrm{rk}_B(b) < \mathrm{rk}_B(1_B) = \mathrm{rk}(B)$. By the induction hypothesis there is a good system for $(B \restriction \tilde{a}, \tilde{a})$.

Case 2. $a/\cong$ has no last element. Hence $\tilde{a} = a/\cong$. Let $(e_\alpha)_{\alpha<\sigma}$ be a stricly increasing sequence, cofinal in $\tilde{a}$. We can suppose that $e_0 = e$, and $e_\beta = \sup\{e_\alpha | \alpha < \beta\}$ for every limit ordinal $\beta < \sigma$ (because $\tilde{a}$ is relatively complete). Let $\alpha < \sigma$ be given. Let $b_\alpha \in B$ be such that $\mathrm{rk}_B(b_\alpha) < \mathrm{rk}_B(1_B)$ and $[e_\alpha, e_{\alpha+1}) \subseteq b_\alpha$. We set $B_\alpha \stackrel{\text{def}}{=} B \restriction [e_\alpha, e_{\alpha+1})$. Since Lemma 2.5, we have $\mathrm{rk}(B_\alpha) \leq \mathrm{rk}(B \restriction b_\alpha) = \mathrm{rk}_B(b_\alpha) < \mathrm{rk}_B(1_B) = \mathrm{rk}(B)$. Applying the induction hypothesis to $(B_\alpha, [e_\alpha, e_{\alpha+1}))$, there is a good system $(B_\alpha, [e_\alpha, e_{\alpha+1}), \mu_\alpha, \psi_\alpha)$. Hence $\psi_\alpha[[e_\alpha, e_{\alpha+1})] = \mu_\alpha$. Let $\mu = \sum_{\alpha<\sigma} \mu_\alpha$, and $\psi = \bigcup\{\psi_\alpha | \alpha < \sigma\}$. We have $\mathrm{Pred}(\mu) = \mu$. We extend ψ in an one-to-one homomorphism $\underline{\psi}$ from $B \restriction \tilde{a}$ into $B(\mu)$: let $b \in B$. We set:

$$\underline{\psi}(b) = \bigcup\{\mu_\alpha | [e_\alpha, e_{\alpha+1}) \subseteq b\} \cup \bigcup\{\psi_\alpha(b \cap [e_\alpha, e_{\alpha+1})) | b \cap [e_\alpha, e_{\alpha+1}) \neq 0_{B_\alpha}, 1_{B_\alpha}\}.$$

We must remark that $\underline{\psi}$ is well defined, because b is a finite union of half-open intervals and thus $\{\alpha < \sigma | [e_\alpha, e_{\alpha+1}) \subseteq b\} \in B(\sigma)$, and the set $\{\alpha < \sigma | b \cap [e_\alpha, e_{\alpha+1}) \neq 0_{B_\alpha}, 1_{B_\alpha}\}$ is finite. Consequently $\underline{\psi}(b)$ is a finite union of half-open intervals of μ, and thus $(B \restriction \tilde{a}, \tilde{a}, \mu, \underline{\psi})$ is a good system. That finishes the proof of Claim 3.6. □

End of the proof of Theorem 1. Let $(a_\zeta/\cong)_{\zeta<\theta}$ be an enumeration of the set of equivalence classes. By Claim 3.6, for $\zeta < \theta$, let $(B \restriction \tilde{a}_\zeta, \tilde{a}_\zeta, \lambda_{\tilde{a}_\zeta}, \psi_{\tilde{a}_\zeta})$ be a good system. Let $\lambda = \sum_{\zeta<\theta} \lambda_{\tilde{a}_\zeta}$. Each $\lambda_{\tilde{a}_\zeta}$ is an interval of λ. Now, let ψ be the function from $\mathrm{At}(B)$ into λ defined by $\psi(a) = \psi_{\tilde{a}_\zeta}(a)$ where $\tilde{a}_\zeta$ is the unique class such that $a \in \mathrm{At}(B) \cap B \restriction \tilde{a}_\zeta$. Let $b \in B$. First, suppose that $\mathrm{rk}_B(b) < \mathrm{rk}_B(1_B)$. There is a finite subset $\{\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1}\}$ of equivalence classes such that $b \subseteq \bigcup\{\tilde{a}_k | k < n\}$, follows from Lemma 3.2. We set $\psi(b) = \bigcup\{\psi_{\tilde{a}_k}(b \cap \tilde{a}_k) | k < n\}$. Now, because $I(B) = I_{\mathrm{rk}(B)}(B)$ is a maximal ideal of B, if $\mathrm{rk}_B(b) = \mathrm{rk}_B(1_B)$, then $\mathrm{rk}_B(-b) < \mathrm{rk}_B(1_B)$, and we set $\psi(b) = -\psi(-b)$. The fact that ψ is a one-to-one homomorphism from B into $B(\lambda)$ is a consequence of the following obvious result:

Claim 3.7. Let B' and B'' be two atomic algebras and ψ be a one-to-one function from $\mathrm{At}(B')$ onto $\mathrm{At}(B'')$. We suppose that, for each $b \in B'$, there is an unique element of B'', denoted by $\underline{\psi}(b)$, such that for every $a \in \mathrm{At}(B')$, we have $a \subseteq b$ if and only if $\psi(a) \subseteq \underline{\psi}(b)$. Then $\underline{\psi}$ is a one-to-one homomorphism from B' into B'', extending ψ. This finishes the proof of Theorem 1. □

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