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GENERIC SPECTRAL PROPERTIES OF MEASURE-PRESERVING MAPS AND APPLICATIONS

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ABSTRACT. Let $\mathscr{K}$ denote the group of all automorphisms of a finite Lebesgue space equipped with the weak topology. For $T \in \mathscr{K}$, let σ_T denote its maximal spectral type.

Theorem 1. *There is a dense G_δ subset $G \subset \mathscr{K}$ such that, for each $T \in G$ and $k(1), \dots, k(l) \in \mathbb{Z}^+$, $k'(1), \dots, k'(l') \in \mathbb{Z}^+$, the convolutions*

$$\sigma_{T^{k(1)}} * \cdots * \sigma_{T^{k(l)}} \quad \text{and} \quad \sigma_{T^{k'(1)}} * \cdots * \sigma_{T^{k'(l')}}$$

are mutually singular, provided that $(k(1), \dots, k(l))$ is not a rearrangement of $(k'(1), \dots, k'(l'))$.

Theorem 1 has the following consequence.

Theorem 2. *$\mathscr{K}$ has a dense G_δ subset $F \subset G$ such that for $T \in F$ the following holds: For any $\mathbf{k}: \mathbb{N} \to \mathbb{Z} - \{0\}$ and $l \in \mathbb{Z} - \{0\}$, the only way that T^l, or any factor thereof, can sit as a factor in $T^{\mathbf{k}(1)} \times T^{\mathbf{k}(2)} \times \cdots$ is inside the ith coordinate σ-algebra for some i with $\mathbf{k}(i) = l$.*

Theorem 2 has applications to the construction of certain counterexamples, in particular nondisjoint automorphisms having no common factors and weakly isomorphic automorphisms that are not isomorphic.

INTRODUCTION

In 1979 Rudolph [R] introduced the notion of minimal self-joinings as the foundation for a machinery that yields a wide variety of counterexamples in ergodic theory. One of the most important of these counterexamples is a pair of finite measure-preserving transformations that are disjoint but have no common factors, answering a question of Furstenburg [Fu]. Another counterexample produced by this machinery is a pair of automorphisms that are weakly isomorphic (i.e., each is a factor of the other) but not isomorphic, answering a question of Sinai [Si]. The notion of simplicity [V, JR] is also sufficient for these constructions. More recently, weakly isomorphic but not isomorphic automorphisms have been constructed using Gaussian processes with spectral measure supported on a Kronecker set [T] and also via group extensions of rotations [L].

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All of these constructions are based on automorphisms T with rather special properties. One manifestation of this special nature is the fact that, with the exception of simplicity, it is known that each of these properties defines a class of automorphisms that is meager with respect to the weak topology on the group of all automorphisms.

One aim of this paper is to show that these constructions, as well as many others in [R], can be based on a much weaker property that is in fact generic (residual in the weak topology). This is Theorem 2. Somewhat surprisingly, it is the genericity of a certain spectral property, Theorem 1, that is responsible for this behaviour. Theorem 1, which is of interest for its own sake, has as special cases a result of Katok [K] and Stepin [St] on the mutual singularity of the convolution powers of the maximal spectral type of an automorphism T, and a result of Choksi and Nadkarni [CoN] on mutual singularity of the maximal spectral types of T^k, $k = 1, 2, \ldots$.

The Katok-Stepin result yields a weaker version of Theorem 2 that is sufficient for the two counterexamples that we have discussed, but other constructions require the full strength of Theorem 2 and hence of Theorem 1. As further applications of Theorem 2, we include a description of the centralizer of $T^{k_1} \times T^{k_2} \times \cdots$, Theorem 3, which in turn shows that most of the examples in §4 of [R] (the exceptions are examples 6,7, and 9) work in this context. In particular, we mention automorphisms with no roots or roots only of prescribed orders. This list is by no means exhaustive. Two examples that certainly do not come out of our work are an automorphism commuting only with its powers and a prime automorphism. In this connection we mention that the property of commuting only with its powers is a meagre property of an automorphism, but it is not known whether this is also the case for primality.

One advantage of using Theorems 1 and 2 for counterexample constructions is that one can make the examples loosely Bernoulli [ORW]. This is because all the constructions are factors of an automorphism that has a power that is a cartesian product of powers of T. That such a cartesian product is loosely Bernoulli for a generic T follows from [Fe] in the case of $T \times T \times \cdots \times T$ and by a similar argument in the general case. We also mention that by [St] there exist measure-preserving diffeomorphisms of the torus satisfying (D) of Theorem 1, so all the counterexamples we have discussed can be made smooth.

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1. Spectral results

We deal throughout with a Lebesgue probability space $(X, \mathscr{B}, \mu)$ and measure-preserving automorphisms $T: (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$. We denote by $\mathscr{K}$ the group of all such automorphisms. $\mathscr{K}$ is endowed with the *weak topology*, defined by

$$T_n \to T \Leftrightarrow \mu(T_n^{-1} A \Delta T^{-1} A) \to 0 \qquad A \in \mathscr{B},$$

which makes $\mathscr{K}$ a Polish (complete separable metric) group. Any property of an automorphism that holds for all T in some dense G_δ subset of $\mathscr{K}$ we will refer to as *generic*.

For $T \in \mathscr{K}$, we denote by σ_T the maximal spectral type of the associated unitary operator U_T on $L_2^0(\mu)$, the L_2-space of functions of mean 0. Our main aim in this section is to prove the following result.

Theorem 1. *For a generic* $T \in \mathscr{K}$, *we have*

(D): *If* $k(1), \ldots, k(l),\ k'(1), \ldots, k'(l') \in \mathbb{Z}^+$ *then*

$$\sigma_{T^{k(1)}} * \cdots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \cdots * \sigma_{T^{k'(l')}}$$

unless the vector $(k'(1), \ldots, k'(l'))$ *is a permutation of* $(k(1), \ldots, k(l))$ (*so, in particular,* $l = l'$).

Theorem 1 has as a special case a result of Katok [K] and Stepin [St] that states that generically the convolution powers σ_T^k, $k = 1, 2, \ldots$ are pairwise mutually singular. Another special case is a result of Choksi and Nadkarni [Co, N] that generically σ_{T^k}, $k = 1, 2, \ldots$, are pairwise mutually singular.

The proof of the Katok-Stepin result is based on the notion of α-weak mixing. According to [K, St] for $\alpha \in [0, 1]$, T is said to be *α-weakly mixing* if there is a sequence $n_i \to \infty$ such that

$$\mu(T^{n_i}A \cap B) \to \alpha\mu(A)\mu(B) + (1-\alpha)\mu(A \cap B) \quad \forall A, B \in \mathscr{B}.$$

(Roughly speaking, along $\{n_i\}$ one has a proportion α of mixing and $1-\alpha$ of rigidity.) We will sometimes speak of α-weak mixing along $\{n_i\}$. In [K, St] it is shown that for each $\alpha \in [0, 1]$, α-weak mixing is generic and that for $\alpha \in (0, 1)$, α-weak mixing implies the desired singularity.

We will use the following strengthening of the notion of α-weak mixing. If $\alpha_1, \ldots, \alpha_K \in [0, 1]$, we will say T is $(\alpha_1, \ldots, \alpha_K)$-*weakly mixing* if there is a sequence $\{n_i\}$ such that for each $k = 1, \ldots, K$ and $A, B \in \mathscr{B}$,

$$\mu(T^{kn_i}A \cap B) \to \alpha_k\mu(A)\mu(B) + (1-\alpha_k)\mu(A \cap B).$$

In other words, each T^k is α_k-weakly mixing along the common sequence $\{n_i\}$.

Theorem 1 follows from the following sequence of lemmas and propositions. The first one is due to Katok [K] and Stepin [St].

Lemma 1.1 ([K, St]). *T is $(1-\alpha)$ weakly mixing along $\{n_i\}$ if and only if for each $f \in L_1(\sigma_T)$, one has*

$$\int_{\mathbb{T}} z^{n_i} f(z)\,d\sigma_T(z) \to \alpha \int_{\mathbb{T}} f(z)\,d\sigma_T(z).$$

(*We consider σ_T as a measure on $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$*).

Lemma 1.2. *Suppose that T is $(1-\alpha_1, \ldots, 1-\alpha_K)$-weakly mixing and $0 < k(1), k(2), \ldots, k(l) \le K$, and let $\sigma = \sigma_{T^{k(1)}} * \cdots * \sigma_{T^{k(l)}}$. Then for any $f \in L_1(\sigma)$, one has*

$$\int z^{n_i} f(z)\,d\sigma(z) \to \alpha_{k(1)}\alpha_{k(2)}\cdots\alpha_{k(l)} \int f\,d\sigma.$$

Lemma 1.3. *Suppose T is $(1-\alpha_1, \ldots, 1-\alpha_K)$-weakly mixing with $\alpha_1, \ldots, \alpha_K \in (0, 1)$ and $\{\log \alpha_1, \ldots, \log \alpha_K\}$ is linearly independent over $\mathbb{Q}$. Then T satisfies the following finite version of* (D):

(D_K): *If* $0 < k(1), \ldots, k(l),\ k'(1), \ldots, k'(l') \le K$, *then*

$$\sigma_{T^{k(1)}} * \cdots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \cdots * \sigma_{T^{k'(l')}},$$

unless $(k'(1), \dots, k'(l'))$ *is a permutation of* $(k(1), \dots, k(l))$.

Lemma 1.4. $\forall K > 0$ *the set of* $T \in \mathscr{K}$ *satisfying* (D_K) *is a* G_δ *subset of* $\mathscr{K}$.

Proposition 1.5. *Suppose* $\beta_1, \dots, \beta_K > 0$ *and* $\sum_{j=1}^K \beta_j = 1$. *For* $1 \le k \le K$, *let* $\alpha_k = \sum_{j|k} \beta_j$. ($j|k$ *means* j *divides* k.) *Then there is a* $T \in \mathscr{K}$ *that is* $(1-\alpha_1, \dots, 1-\alpha_K)$*-weakly mixing.*

Lemma 1.6. $\forall K > 0$ *there exist* $\alpha_1, \dots, \alpha_k \in (0, 1)$ *and* $T \in \mathscr{K}$ *such that* $\{\log, \alpha_1, \dots, \log \alpha_K\}$ *is independent over* $\mathbb{Q}$ *and* T *is* $(1-\alpha_1, \dots, 1-\alpha_K)$ *weakly mixing.*

Proof of Lemma 1.2. This is a strengthening of Lemma 3.9 of [K] (see also [St, Theorem 3.1]), and the proof is very similar. First observe that, since $\sigma_{T^{k(1)}} * \cdots * \sigma_{T^{k(l)}}$ on $\mathbb{T}$ is the image of $\sigma = \sigma_{T^{k(1)}} \times \cdots \times \sigma_{T^{k(l)}}$ on $\mathbb{T}^l$ under the map $(z_1, \dots, z_l) \mapsto z_1 \cdots z_l$, it is enough to prove that for all $F \in L_1(\sigma)$ one has

$$\int_{\mathbb{T}^l} z_1^{n_i} \cdots z_l^{n_i} F \, d\sigma \underset{i}{\to} \alpha_{k(1)} \cdots \alpha_{k(l)} \int F \, d\sigma.$$

Thus, what we need to show is the weak -* convergence of a norm-bounded sequence of functionals in $L_1(\sigma)^*$, so it suffices to consider F of the form

$$F = f_1 \otimes \cdots \otimes f_l, \qquad f_i \in L_1(\sigma_{T^{k(i)}}),$$

since linear combinations of such are dense in $L_1(\sigma)$. But then

$$\int_{\mathbb{T}^l} z_1^{n_i} \cdots z_l^{n_i} f_1 \otimes \cdots \otimes f_l d\sigma = \prod_{j=1}^{l} \int_{\mathbb{T}} z^{n_i} f_j d\sigma_{T^{k(j)}} \to \prod_{j=1}^{l} \left(\alpha_{k(j)} \int_{\mathbb{T}} f_j d\sigma_{T^{k(j)}} \right)$$
$$= \alpha_{k(1)} \cdots \alpha_{k(l)} \int_{\mathbb{T}^l} F d\sigma,$$

by Lemma 1.1. □

Proof of Lemma 1.3. If the measures in question are not mutually singular, we can find a probability measure ρ absolutely continuous with respect to both of them. Then by Lemma 1.2,

$$\int_{\mathbb{T}} z^{n_i} d\rho \to \alpha_{k(1)} \cdots \alpha_{k(l)} \quad \text{and} \quad \int_{\mathbb{T}} z^{n_i} d\rho \to \alpha_{k'(1)} \cdots \alpha_{k'(l')},$$

so

$$\alpha_{k(1)} \cdots \alpha_{k(l)} = \alpha_{k'(1)} \cdots \alpha_{k'(l')}.$$

For $1 \le k \le K$, set $m(k) = \sharp\{j\colon k(j) = k\}$ and $m'(k) = \sharp\{j\colon k'(j) = k\}$. Then the above equation can be rewritten as

$$\prod_{k=1}^{K} (\alpha_k)^{m(k)} = \prod_{K=1}^{K} (\alpha_k)^{m'(k)}.$$

Taking logarithms and using the independence of the $\log \alpha_k$'s, we find that $m(k) = m'(k)$ for each k, which is the desired conclusion. □

Proof of Lemma 1.4. Although it has not yet been made explicit, σ_T is of course not actually a measure, but an equivalence class of measures. As in [CoN] we fix a choice of representative of σ_T for each T by fixing an orthonormal basis $\{\phi_n\}$ of $L_2^0(\mu)$ and setting

$$\sigma_T = \sum_{n=1}^{\infty} 2^{-n} \sigma_{T,\phi_n},$$

where σ_{T,ϕ_n} denotes the spectral measure corresponding to T and ϕ_n, defined by $\hat{\sigma}_{T,\phi_n}(m) = \langle U_T^m \phi_n | \phi_n \rangle$. Note that σ_T is a probability measure. Let $\mathscr{P}(\mathbb{T})$ denote the space of Borel probability measures on $\mathbb{T}$ with the weak-* topology.

Lemma 1.4 follows immediately from the following observations:

(i) For all k the map $T \mapsto T^k$ from $\mathscr{K}$ to $\mathscr{K}$ is continuous.
(ii) The map $T \mapsto \sigma_T$ from $\mathscr{K}$ to $P(\mathbb{T})$ is continuous.
(iii) The map $(\sigma_1, \dots, \sigma_l) \mapsto \sigma_1 * \cdots * \sigma_l$ from $\mathscr{P}(\mathbb{T})^l$ to $\mathscr{P}(\mathbb{T})$ is continuous.
(iv) $\{(\sigma, \tau) \in P(\mathbb{T})^2 : \sigma \perp \tau\}$ is a G_δ subset of $P(\mathbb{T})^2$.

(i), (ii), and (iii) are easy (see also [CoN]). To see (iv), which is similar to Theorem 2 of [Co, N], fix a countable family $\{f_i\} \subset C(\mathbb{T})$ such that $0 \le f_i \le 1$ and $\{f_i\}$ is norm-dense in $\{f \in C(\mathbb{T}) : 0 \le f \le 1\}$. We claim that

$$\text{(v)}\ \{(\sigma, \tau) : \sigma \perp \tau\} = \bigcap_{n>0} \bigcup_i \{(\sigma, \tau) : \sigma(f_i) < 1/n,\ \tau(1 - f_i) < 1/n.\}$$

The left-hand side of (v) is contained in the right-hand side by a standard approximation argument. For the opposite inclusion, if (σ, τ) belongs to the right-hand side then for all $n > 0$ we can find i_n such that $\sigma(f_{i_n}) < 1/n^4$ and $\tau(1 - f_{i_n}) < 1/n^4$. Then

$$\sigma\{f_{i_n} < 1/n^2\} > 1 - 1/n^2$$

and

$$\tau\{f_{i_n} > 1 - 1/n^2\} = \tau\{1 - f_{i_n} < 1/n^2\} > 1 - 1/n^2.$$

Setting $E_n = \{f_{i_n} < 1/n^2\}$ and $F_n = \{f_{i_n} > 1 - 1/n^2\}$, we have $E_n \cap F_n = \varnothing$, $\sigma(E_n) > 1 - 1/n^2$, and $\sigma(F_n) > 1 - 1/n^2$. It follows that

$$\sigma\left(\bigcup_m \bigcap_{m \ge n} E_n\right) = 1 \quad \text{and} \quad \tau\left(\bigcup_m \bigcap_{m \ge n} F_n\right) = 1.$$

Since these two sets are disjoint, it follows that $\sigma \perp \tau$, as desired. □

Proof of Proposition 1.5.

We describe a rank one cutting and stacking construction to achieve the desired properties. We will describe the construction by specifying how copies of the name **a** of a point in the base of the nth tower are concatenated with spacers to form the name **b** of a point in the base of the $(n+1)$st tower. At the same time we produce a $w > 0$ such that for $k = 1, \dots, K$, T^{kw} has approximately the desired amount of mixing and rigidity on sets A and B that are levels of the nth tower. The approximation can be made as close as we want, regardless of what tower we start with, and moreover, the amount of measure added to the space can be made as small as we wish. We use this implicitly in that, when we make an argument on the $(n+1)$st tower, we need to know it represents most of the space. Iterating this construction gives the desired T.

Our choice of spacer sequence will be a "generically random" one in the style of [O] and [R]. This means that we choose a spacer sequence according to a probability distribution on the space of all possible spacer sequences and then argue that with high probability, the choice we have made is suited to our purposes. If $\mathbf{x}$ is a finite sequence of symbols, we denote by $|\mathbf{x}|$ the length of $\mathbf{x}$.

We begin with $\mathbf{a} = 012 \cdots h-1$; that is, the nth tower has h levels, and we consider the partition into those levels. First let

$$\mathbf{c} = \underbrace{\mathbf{aa} \cdots \mathbf{a}}_{C} \equiv \mathbf{a}^C,$$

where C is a large number to be specified later. Next choose M much larger than h, and set

$$w = Ch + M = |\mathbf{a}^C| + M.$$

For $0 \leq s \leq M$ let

$$\mathbf{c}_s = x^s \mathbf{c} x^{M-s}$$

so $|\mathbf{c}_s| = w$ for all s. That is, we place $\mathbf{c}$ in any of the M possible positions in a "window" of length w and add the necessary number of spacers $(x's)$ before and after $\mathbf{c}$ needed to fill up the window.

Now let N be a large number as yet unspecified except for the requirement that $k|N$ for each $k = 1, \dots, K$. For each $k = 1, \dots, K$ choose

$$\mathbf{s}^k = (s_0^k, s_1^k, \dots, s_{N-1}^k) \in \{0, 1, \dots, M\}^N,$$

and let

$$\mathbf{b}_k = (\mathbf{c}_{s_0^k} \cdots \mathbf{c}_{s_{k-1}^k})^{L_k} (\mathbf{c}_{s_k^k} \cdots \mathbf{c}_{s_{2k-1}^k})^{L_k} \cdots (\mathbf{c}_{s_{N-k}^k} \cdots \mathbf{c}_{s_{N-1}^k})^{L_k},$$

where L_k will be specified later. We let $W = [0, 1, \dots, w-1]$, and we refer to each interval of the form $W + tw$, $0 \leq t < NL_k$, as a window in b_k. Finally let

$$\mathbf{b} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_K.$$

We will argue that given $\mathbf{a}$ and $\varepsilon > 0$, if the parameters in the construction of $\mathbf{b}$ are chosen well, then whenever A and B are levels of the nth tower, one has

$$\text{(i)} \qquad \mu(T^{kw} A \cap B) \overset{\varepsilon}{\sim} (1-\alpha_k)\mu(A)\mu(B) + \alpha_k \mu(A \cap B).$$

Notice that by making all L_k sufficiently large once C and M are chosen, we can ensure that kw is so small compared to $|b|$ that (i) is meaningful. Moreover, by adjusting the relative sizes of the L_k, we can ensure that

$$|\mathbf{b}_k|/|\mathbf{b}| \overset{\varepsilon_1}{\sim} \beta_k$$

for any preassigned $\varepsilon_1 > 0$. Thus to establish (i), it will be sufficient to show that given $\varepsilon_2 > 0$, we can choose the parameters so that if A and B denote the union of all levels of $\mathbf{b}$ labelled l and l' respectively $(0 \leq l, l' < h)$, then

$$\mu(T^{kw} A \cap B) \overset{\varepsilon_2}{\sim} \begin{cases} \beta_j \mu(A \cap B) & \text{if } j|k, \\ \beta_j \mu(A)\mu(B) & \text{otherwise.} \end{cases}$$

In other words, conditionally on b_j, T^{kw} is approximately rigid on levels of $\mathbf{a}$ when $j|k$ and approximately mixing otherwise. (i) then follows by summing (ii) over j, keeping in mind the definition of α_k.

Now, $\mathbf{b}_j$ is made up of long blocks of length jwL_j, each of which has period jw, so if L_j is large enough, the rigidity in (ii), when $j|k$, is immediate.

To argue the mixing in (ii), we first establish some notation. When there is no ambiguity, $[m, n]$ will denote an interval in $\mathbb{Z}$. For $E \subset [0, |\mathbf{b}_j| - 1 - kw]$, let π_E denote the frequency distribution on pairs $(l, l') \in [0, h-1]^2$ induced by the double name $(\mathbf{b}_j, S^{kw}\mathbf{b}_j)$, sampled only on E. (Here $S^{kw}\mathbf{b}_j$ denotes the shift of b_j by kw, and we view both $\mathbf{b}_j$ and $S^{kw}\mathbf{b}_j$ as sequences with domain $[0, |\mathbf{b}_j| - 1 - kw]$.) More precisely,

$$\pi_E(l, l') = (|\mathbf{b}_j| - 1 - kw)^{-1}\sharp\{t \in E\colon \mathbf{b}_j(t) = l,\ \mathbf{b}_j(t + kw) = l'\}.$$

Note that π_E is not quite a probability on $\{0, 1, \dots, h-1\}^2$ since spacers also occur (albeit with low frequency) in $\mathbf{b}_j$. When $E = [0, |\mathbf{b}_j| - 1 - kw]$, we write $\pi_E = \pi$.

What we still need to do in order to show (ii) is to argue that, given $\varepsilon_3 > 0$, if the parameters are chosen well, then π is within ε_3 (in total variation norm) of the uniform distribution on $[0, h-1]^2$. By the periodicity of $\mathbf{b_j}$, we may assume $k < j$. We will in fact show something stronger; namely, that restricting our attention to sets E of the form

$$E = \bigcup_{q=0}^{N/j-1} (W + rw + q(jwL_j))$$

with $0 \le r < jL_j$ (i.e., we look at one window in each periodic block in $\mathbf{b}_j$), we already have that π_E is within ε_4 of uniform. Again periodicity allows us to assume that $r \le j-1$.

In the window $W' = W + rw + q(jwL_j)$, we see $\mathbf{c}_{s^j_{r+qj}}$ in $\mathbf{b}_j$ and $\mathbf{c}_{s^j{r'+qj}}$ in $S^{kw}\mathbf{b}_j$, where $r' = r + k(\text{mod } j)$. Thus we see copies of $\mathbf{c}$ separated by a shift of $s^j_{r'+qj} - s^j_{r+qj}$ with $r \neq r'$. If M and hence $d = s^j_{r'+qj} - s^j_{r+qj}$ is sufficiently small compared to $|\mathbf{c}|$, the overlap of these two copies of $\mathbf{c}$ fills most of the window W'. Since $\mathbf{c}$ is a concatenation of many copies of $\mathbf{a}$, it follows that $\pi_{W'}$ is close to the uniform distribution on the subset $\{(l, l')\colon l' = l - d(\text{ mod } h)\}$ of $[0, h-1]^2$. Let us denote this uniform distribution by δ_d.

Now let us suppose that the values $s^j_0, \dots, s^j_{N-1}$ were chosen as a realization of an i.i.d. process $S^j_0, \dots, S^j_{N-1}$ such that the distribution of S^j_i is uniform on $[0, M-1]$. Then the distribution of $S^j_r - S^j_{r'}$(for $r \neq r'$) is the unique tent-shaped distribution on $[-M, M]$, which we will denote by τ_M. Since $S^j_{r'+qj} - S^j_{r+qj}$, $q = 1, \dots, N/j - 1$, are i.i.d., the law of large numbers guarantees that if N is sufficiently large, then with high probability the empirical distribution of the sequence $\{s^j_{r'+qj} - s^j_{r+qj}\}_{q=0}^{N/j-1}$ is as close as we like to τ_M. Thus by the remarks in the previous paragraph, given $\varepsilon_4 > 0$, we can ensure that

$$\pi_E \overset{\varepsilon_4}{\sim} \sum_{d=-M}^{M} \tau_M(d)\delta_d.$$

Now, τ_M is not uniform but we can write τ_M as a convex combination $\tau_M = \sum_{m=0}^{M} a_m\sigma_m$ where σ_m denotes the uniform distribution on $[-m, m]$. Moreover, given $H > 0$, if M is sufficiently large the total weight of $m's$ that are less than H is small then

$$\sum_{m\le H} a_m \le \varepsilon_4.$$

It follows that

$$\pi_E \overset{\varepsilon_4}{\sim} \sum_{d=-M}^{M} \tau_M(d)\delta_d \overset{\varepsilon_4}{\sim} \sum_{d=-M}^{M} \left(\sum_{m\ge H} a_m \sigma_m(d) \right) \delta_d = \sum_{m\ge H} a_m \sum_{d=-M}^{M} \sigma_m(d)\delta_d.$$

If H is sufficiently large compared to h, all the inner sums above will clearly be within ε_4 of uniform on $[0, h-1]^2$, so we conclude that π_E is within $4\varepsilon_4$ of uniform on $[0, h-1]^2$, which finishes the proof.

We now review, after the fact, the logically correct order in which the parameters are chosen. We are given h and $\varepsilon > 0$. These determine how large H must be, and M is then chosen much larger than h. This determines how large C must be to ensure that the window length w is much larger than M. Finally, N and the L_j can be chosen independently of each other. The choice of N ensures mixing and the L_j ensure rigidity. □

Proof of Lemma 1.6. Note that in Lemma 1.5, if we assume only $\beta_1+\cdots+\beta_k \le 1$, then we still get the same conclusion, since we can simply take $\beta_{K+1} = 1 - (\beta_1+\cdots+\beta_K)$ and achieve $(1-\alpha_1, \dots, 1-\alpha_{K+1})$-weak mixing. Note also that, for each $j \le K$, $\beta_1, \dots, \beta_j$ determine $\alpha_1, \dots, \alpha_j$. If we have already chosen $\beta_1, \dots, \beta_j$ so that $\beta_1+\cdots+\beta_j < 1$ and $\{\log \alpha_1, \dots, \log \alpha_j\}$ is independent, then there is still an open interval of values available for β_{j+1}, which means that $\log \alpha_{j+1}$ can range over an open interval of values, and hence we can ensure that $\log \alpha_{j+1}$ is not a rational combination of $\log \alpha_1, \dots, \log \alpha_j$. □

Proof of Theorem 1. Since condition (D) is the intersection of the conditions (D_K), it suffices to prove that D_K is a dense G_δ for each K. In view of Lemma 1.4, we need only prove that D_K is dense and in view of the conjugacy lemma ([H, p. 77]), we need only produce one example of a $T \in D_K$. This is provided by Lemmas 1.3, 1.5, and 1.6. □

Remark. In Lemma 1.5, one could start with an infinite sequence $\beta_1, \beta_2, \dots$, define $\alpha_1, \alpha_2, \dots$ in the same way, and construct a T, that is $(1-\alpha_1, 1-\alpha_2, \dots)$-weakly mixing (with the obvious definition). Moreover, the nature of the construction in Lemma 1.5 is such that one can show, in the manner of [K, St], that $(1-\alpha_1, 1-\alpha_2, \dots)$-weak mixing is itself a generic property. Our approach shows that D is exactly a dense G_δ, rather than merely containing a dense G_δ.

2. Applications

If I is a finite or countable set and $\mathbf{k}\colon I \to \mathbb{Z}$, we denote the cartesian product $\bigotimes_{i\in I} T^{\mathbf{k}(i)}$ acting on (X^I, μ^I) by $T^{\mathbf{k}}$. For $J \subset I$, $\mathscr{B}^J$ will denote the σ-algebra generated by the coordinate projections $x \mapsto x(i)$, $i \in J$. When $I = \{i\}$, we write $B^{\{i\}} = B^i$.

Recall that $T, S \in \mathscr{K}$ are called *disjoint* $(T \perp S)$ if whenever $R \in \mathscr{K}$ and $\mathscr{F}$ and $\mathscr{G}$ are R-invariant σ-algebras such that $R|_{\mathscr{F}} \sim T$ and $R|_G \sim S$, then $\mathscr{F} \perp \mathscr{G}$. By [J] the property $T^k \perp T^{-k}$ for all $k = 1, 2, \dots$ holds for a generic T. Thus if $F \subset \mathscr{K}$ consists of those T for which one has both (D) and $T^k \perp T^{-k}$ for all $k = 1, 2, \ldots$, then F is generic.

Our main result in this section says roughly that if $T \in F$, $\mathbf{k}\colon I \to \mathbb{Z} - \{0\}$, and $l \in \mathbb{Z} - \{0\}$, then $T^{\mathbf{k}}$ and T^l have only the obvious common factors.

Theorem 2. *Suppose that* $T \in F$, $\mathbf{k}\colon I \to \mathbb{Z} - \{0\}$, *and* $l \in \mathbb{Z} - \{0\}$. *Suppose that* $\mathscr{F}$ *and* $\mathscr{G}$ *are invariant* σ*-algebras for* $T^{\mathbf{k}}$ *and* T^l *respectively and that* $T^{\mathbf{k}}|_{\mathscr{F}} \sim T^l|_{\mathscr{G}}$. *Then* $\mathscr{F} \subset \mathscr{B}^i$ *for some* $i \in I$ *such that* $\mathbf{k}(i) = l$.

Proof. We will use the following well-known decomposition of $L_2^0(\mu^I)$:

$$L_2^0(\mu^I) = \bigoplus_{\substack{E \subset I \\ E \text{ finite}}} H_E,$$

where H_E denotes the closed linear span of all functions of the form $\bigotimes_{i \in I} f_i$ with $f_i \in L_2^0(\mu)$ for $i \in E$ and $f_i = 1$ for $i \notin E$. The maximal spectral type of $U_{T^{\mathbf{k}}}$ on H_E is the convolution

$$\prod_{i \in E} \sigma_{T^{\mathbf{k}(i)}}.$$

Our hypothesis imply that the maximal spectral type of the restriction $U_{T^{\mathbf{k}}}|_{L_2^0(\mathscr{F})}$ is absolutely continuous with respect to σ_{T^l} By Theorem 1, σ_{T^l} is singular with respect to the type of $U_{T^{\mathbf{k}}}$ on H_E whenever E contains more than one element, so we conclude that

$$L_2^0(\mathscr{F}) \subset \bigoplus_{i \in I} H_{\{i\}}.$$

This means that for each $A \in \mathscr{F}$,

$$1_A - \mu(A) = \sum_{i \in I} f_i,$$

with $f_i \in H_{\{i\}}$. (Note that $H_{\{i\}} = L_2^0(\mathscr{B}^i)$.) We now want to argue that all but one f_i must vanish. Suppose to the contrary that f_{i_1} and f_{i_2} are both nonzero. Then

$$1_A - \mu(A) = f_{i_1} + f_{i_2} + f,$$

with the functions f_{i_1}, f_{i_2}, and f independent. Let $d(f_{i_1})$, $d(f_{i_2})$, and $d(f)$ denote their distributions, measures on $\mathbb{C}$. Then the distribution of $1_A - \mu(A)$ is the convolution $d(f_{i_1}) * d(f_{i_2}) * d(f)$. The fact that f_{i_1} and f_{i_2} are nonzero with mean zero implies that the supports of $d(f_{i_1})$ and $d(f_{i_2})$ each contain at least two points. It follows that the support of $d(f_{i_1}) * d(f_{i_2}) * d(f)$ contains at least three points, contradicting the fact that $1_A - \mu(A)$ takes only two values.

We have shown that each $A \in \mathscr{F}$ belongs to $\mathscr{B}^i$ for some i. If $A, B \in \mathscr{F}$, $0 < \mu(A)$, $\mu(B) < 1$, $A \in \mathscr{B}^{i_1}$, and $B \in \mathscr{B}^{i_2}$ with $i_1 \neq i_2$, then $A \cap B \in \mathscr{F}$ but $A \cap B$ does not belong to any $\mathscr{B}^j$, a contradiction. Thus we have $\mathscr{F} \subset \mathscr{B}^i$ for some i. Finally, since $\sigma_{T^{-k}} = \sigma_{T^k}$ and $\sigma_{T^{k_1}} \perp \sigma_{T^{k_2}}$ for $|k_1| \neq |k_2|$, we must have $|\mathbf{k}(i)| = |l|$. Also, since $T^{-\mathbf{k}(i)} \perp T^{\mathbf{k}(i)}$, which precludes $T^{\mathbf{k}(i)}$ and $T^{-\mathbf{k}(i)}$ having any common factors, we must have $\mathbf{k}(i) = l$. □

Theorem 2 allows us to describe the (not necessarily invertible) measure-preserving transformations commuting with $T^{\mathbf{k}}$. We can make the description more explicit by assuming another generic property of T. If T admits a sufficiently fast cyclic approximation in the sense of [K, St] then by [ACaSc] any homomorphism S commuting with T is a weak limit of powers of T, and in particular, this forces S to be invertible. It is easy to see that for a generic T, T^k will have such a cyclic approximation for each $k \in \mathbb{Z} - \{0\}$. Thus if we denote by $C(T)$ the set of homomorphisms commuting with T, we see that the set F' of T such that $C(T^k) = \mathrm{cl}\{T^n \colon n \in \mathbb{Z}\}$ for all $k \neq 0$ is generic. In particular, $C(T^k)$ is a group. Any T such that $C(T)$ is a group is called *coalescent* [HhP, N]. Coalescence is equivalent to the nonexistence of proper invariant σ-algebras $\mathscr{F}$ such that $T|_{\mathscr{F}} \sim T$.

If J and I are finite or countable and $\pi\colon J \to I$ is an injection, let us denote by U_π the measure-preserving map $(X^I, \mu^I) \to (X^J, \mu^J)$ defined by

$$(U_\pi(x))(j) = x(\pi(j)).$$

If $S\colon J \to \mathscr{K}$, we denote by $\bigotimes S$ the cartesian product $\bigotimes_{j\in J} S(j)$ on (X^J, μ^J).

Corollary 3. *Suppose that* $T \in F \cap F'$.

(a) *Suppose that* $\mathbf{k};\ I \to \mathbb{Z} - \{0\}$ *and* $\mathbf{l};\ J \to \mathbb{Z} - \{0\}$. *Then any homomorphism* V *from* $T^{\mathbf{k}}$ *to* $T^{\mathbf{l}}$ *is of the form* $(\bigotimes S)U_\pi$ *for some injection* $\pi\colon J \to I$ *and* $S\colon J \to \mathrm{cl}\{T^n\colon n \in \mathbb{Z}\}$.

(b) *If* $\mathbf{k}\colon I \to \mathbb{Z} - \{0\}$ *then the centralizer of* $T^{\mathbf{k}}$ *is the semigroup of maps of the form* $(\bigotimes S)U_\pi$ *where* $\pi\colon I \to I$ *is an injection such that* $\mathbf{k}$ *is constant on orbits of* π *and* $S\colon I \to \mathrm{cl}\{T^n\colon n \in \mathbb{Z}\}$. *In particular, if* I *is finite, the centralizer of* $T^{\mathbf{k}}$ *is a group.*

Proof. (a) For $j \in J$, $V^{-1}(\mathscr{B}^j)$ is an invariant σ-algebra for $T^{\mathbf{k}}$, and $T^{\mathbf{k}}|_{V^{-1}(\mathscr{B}^j)}$ is isomorphic to $T^{\mathbf{l}(j)}$. By Theorem 2 this means that $V^{-1}(\mathscr{B}^j) \subset \mathscr{B}^i$ for some i with $\mathbf{k}(i) = \mathbf{l}(j)$. Denote $i = \pi(j)$. Now $T^{\mathbf{k}}|_{\mathscr{B}^i} \sim T^{\mathbf{k}(i)}$, and $T^{\mathbf{k}}|_{V^{-1}(\mathscr{B}^j)} \sim T^{\mathbf{l}(j)} = T^{\mathbf{k}(i)}$, so by coalescence of $T^{\mathbf{k}(i)}$, we conclude that $V^{-1}(\mathscr{B}^j) = \mathscr{B}^i$. Since $V^{-1}(\mathscr{B}^{j_1}) \perp V^{-1}(\mathscr{B}^{j_2})$ for $j_1 \neq j_2$, we must have $\pi(j_1) \neq \pi(j_2)$, and it follows that V has the desired form.

(b) follows immediately from (a). □

Thus, for example, for $T \in F \cap F'$, $C(T \times T^{-1}) = C(T) \times C(T)$. In a similar vein, one has the following result.

Proposition 4. *Suppose that* $T \perp S$ *and both* σ_T *and* σ_S *are singular with respect to* $\sigma_T * \sigma_S$. *Then* $C(T \times S) = C(T) \times C(S)$.

Proof. Same as the proof of Theorem 2 and Corollary 3. □

We conclude by mentioning some counterexamples that follow from Corollary 3. These constructions can all be found in [R]; our aim here is simply to point out that many of them can be carried out under relatively weak hypotheses on T, namely, those of Theorem 2 or 3.

Example 1. Nondisjoint $T, S \in \mathscr{K}$ that have no common factors.

Take T as in Theorem 2, and let S be the symmetric cartesian square $T^{2\odot}$ of T, namely, the restriction of $T \times T$ to the σ-algebra $\mathscr{B}^{2\odot}$ of sets in $\mathscr{B} \times \mathscr{B}$ that are invariant under the map $(x, y) \mapsto (y, x)$. T and $T^{2\odot}$ are never disjoint, as they are both factors of $T \times T$ and the σ-algebras in question, namely, $\mathscr{B}^1$ and $\mathscr{B}^{2\odot}$, are not independent. On the other hand, if $\mathscr{F} \subset \mathscr{B}^{2\odot}$ is a $T^{2\odot}$ invariant σ-algebra such that $T^{2\odot}|_{\mathscr{F}} = (T \times T)|_{\mathscr{F}}$ is isomorphic to a factor of T, then by Theorem 2, $\mathcal{F}$ is contained in, say, $\mathcal{B}^1$. But $\mathcal{B}^1 \cap \mathcal{B}^{2\odot}$ is obviously trivial so $\mathcal{F}$ is trivial.

Example 2. Weakly isomorphic R, $S \in \mathcal{K}$ that are not isomorphic.

Take T as in Theorem 2, and let $R = T \times T \times \cdots$ acting on $X^{\{1,2,\dots\}}$ and $S = T^{2\odot} \times (T \times T \times \cdots)$ acting on the σ-algebra $\mathcal{B}^{2\odot} \times \mathcal{B}^{\{3,4,\dots\}}$ in $X^{\{1,2\}} \times X^{\{3,4,\cdots\}}$. R and S are obviously weakly isomorphic. If V is an isomorphism from R to S, then for $i = 1, 2, \dots$, $V(\mathcal{B}^i)$ is a $T^{2\odot} \times (T \times T \times \cdots)$ invariant subalgebra of $\mathcal{B}^{2\odot} \times \mathcal{B}^{\{3,4,\dots\}}$ such that $[T^{2\odot} \times (T \times T \times \cdots)]|_{V(\mathcal{B}^i)} = [T \times T \times (T \times T \times \cdots)|_{V(\mathcal{B}^i)}$ is isomorphic to T. By Theorem 2, $V(\mathcal{B}^i) \subset \mathcal{B}^j$, and as in Example 1, we see that $j = 1$ or 2 is impossible. Thus we find that $V(\mathcal{B}^i) \subset \mathcal{B}^{\{3,4,\dots\}}$ for all i, contradicting the fact that V is an isomorphism.

Example 3. $S \in \mathcal{K}$ with no roots, roots of specified orders only, or many nonisomorphic roots of a given order.

[R] contains several example along these lines. We leave it to the reader to check that the same examples work here.

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