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THE ANALYTIC RANK OF A C^*-ALGEBRA

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ABSTRACT. A concept of rank for unital C^*-algebras, which reduces to the dimension of the spectrum in the case of separable abelian C^*-algebras, is introduced. This rank has properties somewhat similar to the stable rank, but it appears to be easier to compute in many cases. It is very well behaved with respect to tensor products and crossed products with countable discrete abelian groups.

1. INTRODUCTION

A number of concepts of rank for a C^*-algebra have been proposed recently. An important part of the motivation for introducing them is to have an analogue for a C^*-algebra of the dimension for a topological space. In [10] the topological stable rank of a C^*-algebra was introduced by Rieffel in the context of K-theory. It was later shown to be identical to the Bass stable rank [5]. Following along similar lines to those set down in [10], Brown and Pedersen [2] proposed the real rank as a useful analogue of topological dimension. An interesting point in this case is that a C^*-algebra is of real rank zero if and only if it satisfies Pedersen's Condition FS [8].

In this paper we shall present another concept of rank for a C^*-algebra, the analytic rank, which reduces to the dimension of the spectrum in the abelian case (as does the real rank [2]). In [6] analytic subalgebras of real abelian normed algebras are defined and analysed. In the case that $A = C(\Omega)$, where Ω is a compact Hausdorff space, it turns out that a closed real subalgebra L of A_{sa} is analytic if and only if L contains the unit 1 and $b^2 \in L$ implies that $b \in L$ for each $b \in A_{sa}$. If A_{sa} is the smallest analytic subalgebra containing a set S, then S is said to be an analytic base for A_{sa}, and the analytic rank of A_{sa} is the least cardinal number such that A_{sa} has an analytic base of that cardinality. The reason for our interest in these concepts is that if Ω is metrisable, then the analytic rank of A_{sa} is equal to the dimension of Ω [6, Proposition 10.4.21]. It is natural to try to extend the ideas involved here to the case of nonabelian C^*-algebras. This leads to a rank function that has some very nice properties. For instance, it behaves with respect to crossed products and tensor products in the way one would like, and the proofs of this are very easy (see §2). It also

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appears to give the "right" number for the rank in some natural situations, for instance, if $\mathbf{F}_n$ is the free group on n generators, the analytic rank of $C^*(\mathbf{F}_n)$ is n (see Example 2.8).

2. Analytic C^*-subalgebras

Let A be a unital C^*-algebra. Motivated by the considerations presented in the introduction we say that a C^*-subalgebra B is an *analytic* C^*-subalgebra of A if B contains the unit of A and for all $a \in A_{\text{sa}}$ such that $a^2 \in B$, necessarily $a \in B$. Clearly A is an analytic C^*-subalgebra of itself, and the intersection of analytic C^*-subalgebras of A is again one. Hence if S is a subset of A, there is a smallest such algebra containing S. We denote it by $A[S]$, and call it the analytic C^*-subalgebra *generated* by S. If $S \subseteq A_{\text{sa}}$ and $A[S] = A$, we say S is an *analytic base* of A.

2.1. *Remark.* If B is an analytic C^*-subalgebra of A, then B contains the symmetries of A, and therefore, the projections of A (since if p is a projection then $2p - 1$ is a symmetry). It follows from Lemma 1 of [8] that for any projection p of A and any $a \in A$ we have $pa - ap \in B$. It also follows that B contains all finite-spectrum normal elements of A.

The field $\mathbf{C}$ is an analytic C^*-subalgebra of A if the only projections of A are 0 and 1. For if $a \in A_{\text{sa}}$ and $a^2 = \lambda$ for some nonzero scalar λ, then $a/\sqrt{\lambda}$ is a symmetry, so $(a/\sqrt{\lambda} + 1)/2$ is a projection, and therefore a scalar. Hence $a \in \mathbf{C}$. Conversely, if $\mathbf{C}$ is analytic in A, the only projections of A are 0 and 1, by Remark 2.1.

In particular if $A = C(\Omega)$, where Ω is a compact Hausdorff space, then $\mathbf{C}$ is an analytic C^*-subalgebra of A if and only if Ω is connected.

For A an arbitrary unital C^*-algebra we define $\operatorname{rank}(A)$, the *analytic rank* of A, to be $+\infty$ if A admits no finite analytic base and we set $\operatorname{rank}(A) = n$ if A admits an analytic base of n elements but none of smaller cardinality. Of course, if S is a subset of A_{sa} generating A as a C^*-algebra, then it is trivially an analytic base of A, and therefore the cardinality of S gives an upper bound for the analytic rank of A.

If Ω is a compact metric space, then it follows from classical dimension theory that $\operatorname{rank}(C(\Omega)) = \dim(\Omega)$, the dimension of Ω (cf. the discussion in the introduction).

If A is a simple unital C^*-algebra, $A \neq \mathbf{C}$, then A is of zero analytic rank if and only if A has nonscalar projections. This is immediate from Proposition 2 of [8] and the preceding remarks.

2.2. **Example.** If $n \geq 2$ and O_n is the corresponding (simple) Cuntz algebra [3] then $\operatorname{rank}(O_n) = 0$.

2.3. **Example.** If A_θ is an irrational rotation algebra then $\operatorname{rank}(A_\theta) = 0$, since A_θ is simple and contains nonscalar projections.

2.4. **Example.** If $\mathbf{F}_2$ is the free group on two generators, then the simple C^*-algebra $C^*_{\text{red}}(\mathbf{F}_2)$ has no nonscalar projections [9], so its analytic rank is positive. The following result implies that $\operatorname{rank}(C^*_{\text{red}}(\mathbf{F}_2)) = 1$ or 2 since $C^*_{\text{red}}(\mathbf{F}_2)$ is generated by a pair of unitaries.

2.5. **Theorem.** *Suppose that A is a unital C^*-algebra generated (as a C^*-algebra) by n partial isometries. Then* $\operatorname{rank}(A) \leq n$.

Proof. To prove the result it suffices to show that if B is an analytic C^*-subalgebra of A and u is a partial isometry of A such that B contains its real part $\mathrm{Re}(u)$ then $u \in B$. (For then if $u_1, \ldots, u_n$ are partial isometries generating A, it will follow that $\mathrm{Re}(u_1)\ldots, \mathrm{Re}(u_n)$ form an analytic base for A.) Let v, w be the real and imaginary parts of u. Then

$$v^2 = \frac{1}{4}(u^2 + u^{*2} + uu^* + u^*u).$$

Also, uu^*, u^*u are projections, so $u^2 + u^{*2} \in B$ since $v^2, uu^*, u^*u \in B$. However,

$$w^2 = -\frac{1}{4}(u^2 + u^{*2} - uu^* - u^*u),$$

so $w^2 \in B$, and therefore $w \in B$ since B is an analytic C^*-subalgebra of A. Hence $u = v + iw \in B$. □

2.6. **Example.** Suppose $\mathbf{A}$ is the *Toeplitz algebra*, that is, the C^*-algebra generated by all Toeplitz operators T_φ with continuous symbol φ on the unit circle $\mathbf{T}$. It is well known [4] that the commutator ideal of $\mathbf{A}$ is $\mathbf{K} = K(H^2)$, the C^*-algebra of compact operators on the Hardy space H^2, and that $\mathbf{A}/\mathbf{K} = C(\mathbf{T})$, where $\mathbf{T}$ is the unit circle. Also, $\mathbf{A}$ is generated by the unilateral shift on the standard orthonormal basis of H^2. We claim that $\mathrm{rank}(\mathbf{A}) = 1$. This will follow from the preceding theorem if we show that $B = \mathbf{K} + \mathbf{C}1$ is an analytic C^*-subalgebra of $\mathbf{A}$. To see this, suppose that $v \in A_{\mathrm{sa}}$ and $v^2 \in B$. Then $v = T_\varphi + w$ for some $\varphi \in C(\mathbf{T})$ and $w \in \mathbf{K}$, so $v^2 = T_{\varphi^2} + w'$ for some $w' \in \mathbf{K}$. Hence T_{φ^2} is a scalar operator, so φ^2 is a constant function, and therefore so is φ, since $\mathbf{T}$ is connected. Hence $v \in B$.

2.7. **Theorem.** *Let $\varphi: A \to A'$ be a surjective $*$-homomorphism of unital C^*-algebras A and A'.*

(1) The map $B \mapsto \varphi(B)$ is a bijective correspondence of the analytic C^*-subalgebras B of A that contain the kernel of φ onto the analytic C^*-subalgebras of A'.

(2) We have the inequality

$$\mathrm{rank}(A') \leq \mathrm{rank}(A).$$

Proof. The proof of condition (1) is completely routine. To see (2) let S be an analytic base for A. If B is the analytic C^*-subalgebra of A' generated by $\varphi(S)$, then $\varphi^{-1}(B)$ is an analylic C^*-subalgebra of A containing S, so $\varphi^{-1}(B) = A$. Hence $B = \varphi(\varphi^{-1}(B)) = A'$, so $\varphi(S)$ is an analytic base for A'. It follows that $\mathrm{rank}(A') \leq \mathrm{rank}(A)$. □

2.8. **Example.** If n is a positive integer and $\mathbf{F}_n$ is the free group on n generators then $\mathrm{rank}(C^*(\mathbf{F}_n)) = n$. To see this note that by Theorem 2.5, $\mathrm{rank}(C^*(\mathbf{F}_2)) \leq n$, since $C^*(\mathbf{F}_n)$ is generated by n unitaries, and since $C(\mathbf{T}^n)$ is a quotient of $C^*(\mathbf{F}_n)$, therefore by Theorem 2.7 $\mathrm{rank}(C^*(\mathbf{F}_n)) \geq \mathrm{rank}(C(\mathbf{T}^n)) = \dim(\mathbf{T}^n) = n$.

2.9. **Theorem.** *Let A_1, A_2 be the unital C^*-algebras and set $A = A_1 \oplus A_2$. Then*

$$\mathrm{rank}(A) = \max\{\mathrm{rank}(A_1), \mathrm{rank}(A_2)\}.$$

Proof. Since A_1, A_2 are quotients of A, their ranks are not greater than the rank of A, by Theorem 2.7. Let B be an analytic C^*-subalgebra of A. Because, $(1 - 1)$ is a symmetry of A it belongs to B, and therefore for all $(a_1, a_2) \in A$ we have $(a_1, a_2) \in B$ if and only if $(a_1, 0)$ and $(0, a_2)$ belong to B. It follows easily from this that if $\pi_i \colon A \to A_i$ is the projection of A onto A_i $(i = 1, 2)$, then $\pi_1(B)$ and $\pi_2(B)$ are analytic C^*-subalgebras of A_1 and A_2 respectively.

Let $n = \max\{\mathrm{rank}(A_1), \mathrm{rank}(A_2)\}$. Then we may choose analytic bases $a_1^1, \dots, a_n^1$ and $a_1^2, \dots, a_n^2$ for A_1 and A_2 respectively. Set $a_j = (a_j^1, a_j^2)$ for $j = 1, \dots, n$. If B is the analytic C^*-subalgebra of A generated by $a_1, \dots, a_n$ then $\pi_i(B)$ is an analytic C^*-subalgebra of A_i containing the analytic base $a_1^i, \dots, a_n^i$, so $\pi_i(B) = A_i$ $(i = 1, 2)$. Hence $B = A$, and $a_1, \dots, a_n$ is an analytic base for A. Therefore $\mathrm{rank}(A) \le n$. □

2.10. **Theorem.** *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of C^*-subalgebras of a unital C^*-algebra A containing the unit of A and such that $\bigcup_{\lambda\in\Lambda} A_\lambda$ generates A. Then*

$$\mathrm{rank}(A) \le \sum_{\lambda\in\Lambda} \mathrm{rank}(A_\lambda).$$

Proof. Let S_λ be an analytic base for A_λ $(\lambda \in \Lambda)$, and let B be the analytic C^*-subalgebra of A generated by $\bigcup_{\lambda\in\Lambda} S_\Lambda$. To prove this result it clearly suffices to show that $B = A$. The intersection $B \bigcap A_\Lambda$ is an analytic C^*-subalgebra of A_λ containing the analytic base S_λ, so $B \cap A_\lambda = A_\lambda$. Hence B contains each A_λ, and therefore since $\bigcup_\lambda A_\lambda$ generates A we have $B = A$. □

If A_1 and A_2 are C^*-algebras we denote their spatial tensor product by $A_1 \otimes_* A_2$.

2.11. **Corollary.** *Let A_1 and A_2 be unital C^*-algebras and $A = A_1 \otimes_* A_2$. Then*

$$\mathrm{rank}(A) \le \mathrm{rank}(A_1) + \mathrm{rank}(A_2).$$

Proof. The algebra A is generated by the union of the C^*-subalgebras $A_1 \otimes 1$ and $1 \otimes A_2$, which are $*$-isomorphic to A_1 and A_2 respectively. □

2.12. **Corollary.** *Let (A, G, α) be a C^*-dynamical system where A is a unital C^*-algebra and G is a countable discrete abelian group. Then*

$$\mathrm{rank}(A \times_\alpha G) \le \mathrm{rank}(A) + \dim(\hat{G}),$$

where $\hat{G}$ is the Pontryagin dual of G.

Proof. We may suppose that $n = \dim(\hat{G}) < \infty$. Since G is countable, $\hat{G}$ is metrisable. Observe that $C^*(G) = C(\hat{G})$, so $C^*(G)$ has analytic rank n. There is a canonical $*$-homomorphism from $C^*(G)$ to $A \times_\alpha G$ whose range we denote by B. By Theorem 2.7 $\mathrm{rank}(B) \le \mathrm{rank}(C^*(G))$. Since the C^*-algebra $A \times_\alpha G$ is generated by the C^*-subalgebras A and B it follows from Theorem 2.10 that $\mathrm{rank}(A \times_\alpha G) \le \mathrm{rank}(A) + n$. □

2.13. **Corollary.** *Let α be an automorphism of a unital C^*-algebra A. Then*

$$\mathrm{rank}(A \times_\alpha \mathbf{Z}) \le \mathrm{rank}(A) + 1.$$

If in Corollary 2.12 the group G is generated by n elements then $C^*(G)$ is generated by n corresponding unitaries, so by Theorem 2.5 $\dim(\hat{G}) = \mathrm{rank}(C^*(G)) \le n$.

The results for the stable and real ranks corresponding to the preceding three corollaries are rather more difficult to obtain, or are not known, see [2, 10].

3. C^*-Algebras of analytic rank zero

C^*-algebras of analytic rank zero exist in great abundance. For instance, every von Neumann algebra is one, as is every unital AF-algebra. More generally, every unital C^*-algebra satisfying Condition FS [8] is of zero analytic rank. The reason is that these algebras are the closed linear span of their projections. The class of algebras of zero analytic rank, however, is larger even than these examples suggest, as we shall see presently.

The following result shows that in some respects the analytic rank behaves quite differently from the stable and real ranks. If A is of finite real and stable rank then the corresponding rank of $M_n(A)$ tends to get small quickly, that is, for rather low values of n. However for $n > 1$ the analytic rank of $M_n(A)$ is always zero!

3.1. **Theorem.** *If A is a unital C^*-algebra and $n > 1$, then $M_n(A)$ is of zero analytic rank.*

Proof. We have to show that if B is an analytic C^*-subalgebra of $M_n(A)$ then $B = M_n(A)$. Identify $M_n(A)$ with $M_n \otimes A$, and let e_{ij} be the matrix in $M_n(\mathbf{C})$ all of whose entries are 0 except for the ij-entry, which is 1. If $a \in A$ then $b = (e_{ij} \otimes a)(e_{jj} \otimes 1) - (e_{jj} \otimes a)(e_{ij} \otimes a)$ belongs to B since $e_{jj} \otimes 1$ is a projection (cf. Remark 2.1). Observe that $b = e_{ij} \otimes a$ if $i \neq j$. Since, in particular, $e_{ji} \otimes 1 \in B$, therefore $b(e_{ji} \otimes 1) = e_{ii} \otimes a \in B$. Thus the elements $e_{ij} \otimes a$ belong to B for all indices i, j, and these elements linearly span $M_n(\mathbf{C}) \otimes A$, so $B = M_n(\mathbf{C}) \otimes A$. □

3.2. **Theorem.** *If C is a uniformly hyperfinite algebra, $C \neq \mathbf{C}$, then $C \otimes_* A$ is of analytic rank zero for any unital C^*-algebra A.*

Proof. The C^*-algebra C can be written $C = (\bigcup_{n=1}^{\infty} C_n)^-$ where each C_n is a simple C^*-subalgebra of finite dimension greater than 1. Hence the union $\bigcup_{n=1}^{\infty} C_n \otimes A$ generates $C \otimes_* A$. Therefore (using Theorem 2.10) to prove the result we may suppose that C is a simple finite-dimensional C^*-algebra of dimension greater than 1, that is, $C = M_n(\mathbf{C})$, for some integer $n > 1$. In this case $C \otimes_* A = M_n(C)$ is of zero analytic rank by Theorem 3.1. □

3.3. **Example.** As we observed above, if a C^*-algebra is the closed linear span of its projections then its analytic rank is zero. The converse is not true, even if the algebra is simple. Blackadar [1] has constructed a simple unital C^*-algebra A for which $M_2(A)$ (which is necessarily of zero analytic rank) is not the closed linear span of its projections. This example serves to illustrate another point, namely, that rank is not monotonic even on hereditary C^*-subalgebras. The algebra A has no nonscalar projections, so $\operatorname{rank}(A) > 0$. If

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then A is clearly *-isomorphic to $B = pM_2(A)p$. Thus B is an example of a hereditary C^*-subalgebra of a zero analytic rank C^*-algebra that is not itself of zero analytic rank.

3.4. **Theorem.** *If a unital C^*-algebra A is the direct limit of a sequence $(A_n)_{n=1}^{\infty}$ of unital C^*-algebras all of zero analytic rank, then A is of zero analytic rank.*

Proof. If $\varphi^n: A_n \to A$ denotes the natural map for each positive integer n, then A is the closure of $\bigcup_{n=1}^{\infty} \varphi^n(A_n)$. By Theorem 2.7 $\varphi^n(A_n)$ is of zero analytic rank, since A_n is. Hence by Theorem 2.9 the C^*-subalgebra $B_n = \varphi^n(A_n) + \mathbf{C}1$ is of zero analytic rank, where 1 is the unit of A (either $B_n = \varphi^n(A_n)$ or B_n is the direct sum of $\varphi^n(A_n)$ and $\mathbf{C}$). Since A is generated by $\bigcup_{n=1}^{\infty} B_n$, therefore by Theorem 2.10 $\operatorname{rank}(A) \le \sum_{n=1}^{\infty} \operatorname{rank}(B_n) = 0$. □

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