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POSITIVE SOLUTIONS OF $\Delta u + K(x)u^p = 0$ WITHOUT DECAY CONDITIONS ON $K(x)$

XINGBIN PAN

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ABSTRACT. This paper deals with the existence of positive solutions of the nonlinear elliptic equation $\Delta u + K(x)u^p = 0$ in R^n with $n \geq 3$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, where $K(x)$ does not decay at ∞. The existence of classical positive solutions and singular positive solutions is proved under the hypothesis that K is radial symmetric, $K(r) = 1 + H(r)$ is a perturbation of the constant 1, and $H(r)$ satisfies some conditions.

Since the publication of [N1], there has been lots of research on the elliptic equation in R^n with $n \geq 3$:

$$(1) \qquad \Delta u + K(x)u^p = 0.$$

For a survey see [N2]. When $K(x)$ decays at ∞, one has a good understanding of equation (1), while for $K(x)$ without decay conditions and $1 < p < \frac{n+2}{n-2}$, the story is incomplete.

In this paper we consider the case $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and $0 < a \leq K(x) \leq b < \infty$. In general, equation (1) has no classical positive solution for such $K(x)$. It is well known that for $K(x) \equiv K_0 > 0$, equation (1) has only singular positive solutions for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and has no positive solution for $1 < p \leq \frac{n}{n-2}$. Little is known for $K(x)$ not equal to the constant.

The main objective of this paper is to establish some sufficient and necessary conditions for existence of classical positive solutions. The gap between sufficiency and necessity seems large, and complete answers remain open.

In this paper we always assume that $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and K is a radial function in R^n satisfying

$$(2) \qquad K(r) \in C^2[0, \infty), \qquad 0 < a \leq K(r) \leq b < \infty,$$

where $r = |x|$. So that $K'(0) = 0$. We begin our analysis by recalling some known results.

Lemma 1. *Assume condition* (2) *holds.*

(i) *For* $1 < p \leq \frac{n}{n-2}$, *equation* (1) *has no positive solution in* R^n *or in* $R^n \backslash B_R$.

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(ii) *For* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, *if* $u = u(r)$ *is a radial positive solution of equation* (1) *in* R^n *or in* $R^n \backslash B_R$, *then*

$$C_1 r^{2-n} \leq u(r) \leq C_2 r^{-2/(p-1)},$$
$$|u'(r)| \leq C_3 r^{-2/(p-1)-1},$$

for r *large.*

(iii) *For* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, *if* $u = u(r)$ *is a radial positive solution of equation* (1) *in* R^n *or in* B_R, *which tends to* ∞ *at* $r = 0$, *then for* $0 < r < R$,

$$u(r) \leq C_4 r^{-2/(p-1)} \quad \text{and} \quad |u'(r)| \leq C_5 r^{-2/(p-1)-1}.$$

For the proof see [N1, Theorem 3.35, 3.41; and NS, Lemma 2.16].

The following lemma is used to prove existence of singular solutions for equation (1). By a singular solution we mean a positive classical solution in $R^n \backslash \{0\}$, which tends to zero at ∞ and tends to ∞ at the origin.

Lemma 2. *For* $n \geq 3$ *and* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, *the only positive solutions of the following equation in* R^n

$$\Delta w + w^p = 0 \tag{3}$$

are a family of singular radial solutions with decay $r^{-(n-2)}$ *at* $r = +\infty$ *and a singular solution* $w_s(r) = L r^{-2/(p-1)}$, *where*

$$L = \left[\frac{2(n-2)}{(p-1)^2}\left(p - \frac{n}{n-2}\right)\right]^{1/(p-1)}.$$

See [GS, Appendix A].

The following Pohozaev identity concerning the solution of equation

$$\Delta u + f(x, u) = 0 \tag{4}$$

was presented in [DN, Lemma 3.7].

Lemma 3. *Let* Ω *be a bounded smooth domain in* R^n *and* u *be a classical solution of* (4). *Then we have*

$$\begin{aligned} &\int_\Omega \left[nF(x, u) - \frac{n-2}{2} u f(x, u) + x \cdot F_x(x, u)\right] dx \\ &= \int_{\partial\Omega} \left[(x \cdot \nabla u)\frac{\partial u}{\partial \nu} - (x \cdot \nu)\frac{|\nabla u|^2}{2} + (x \cdot \nu) F(x, u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu}\right] ds, \\ &\qquad \text{where } F(x, u) = \int_0^u f(x, t)\, dt, \end{aligned} \tag{5}$$

F_x *is the gradient of* F *with respect to* x, *and* ν *is the unit outer normal of* $\partial\Omega$.

Our first result concerns existence of singular solutions of equation (1).

Theorem 1. *Assume* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ *and* $K(r)$ *satisfies condition* (2) *and*

$$\left(\frac{n}{p+1} - \frac{n-2}{2}\right) K(r) + \frac{1}{p+1} r K'(r) \geq 0. \tag{6}$$

Then equation (1) *has a singular solution* $u(r)$ *with singularity* $r^{-2/(p-1)}$ *at zero and with decay* $r^{-(n-2)}$ *at* ∞.

Proof. Let $v_0(r)$ be the radial ground state of the scalar field equation

$$\Delta v_0 - v_0 + v_0^p = 0.$$

From [GNN, Theorem 2],

$$\lim_{r\to\infty} r^{(n-1)/2} e^r v_0(r) = \mu > 0.$$

Let $v(r) = a^{-1/(p-1)} v_0(r)$, then

$$\Delta v - v + av^p = 0. \tag{7}$$

Let $w_0(r)$ be the singular radial solution of equation (3) with decay $r^{-(n-2)}$ at $r = +\infty$. Let $w(r) = b^{-1/(p-1)} w_0(r)$. Then

$$\Delta w + bw^p = 0. \tag{8}$$

From condition (2) we have

$$\Delta w + K(r)w^p \le 0 \quad \text{and} \quad \Delta v + K(r)v^p \ge 0.$$

From the asymptotic behaviors of $v(r)$ and $w(r)$ we have $v(r) < w(r)$ for r large. Without loss of generality we assume b is large enough so that there is a $R > 0$ such that $v(R) = w(R)$ and $v(r) < w(r)$ for $r > R$. Therefore (v, w) is a couple of sub- and super-solutions of the following equation

$$u'' + \frac{n-1}{r}u' + K(r)u^p = 0, \tag{9}$$

which is the radial version of equation (1). From the proof of [N1, Theorem 2.10] or using [P, Lemma 6] we conclude that equation (9) has a C^2 solution $u(r)$ for $r > R$ and $v(r) \le u(r) \le w(r)$. As in the proof of [P, Theorem 1] we have $u'(r) < 0$ for $r > R$.

Now we continue this solution $u(r)$ backwards into $r \le R$. Set $\Sigma = \{\rho > 0 : u(r)$ satisfies (9) in $(\rho, +\infty)$ and $u'(r) < 0\}$, and $\tau = \inf \Sigma$.

Assume $\tau > 0$. As in the proof of [P, Theorem 1] we know $u(\tau) = \lim_{r\to\tau+0} u(r)$ exists and $u'(\tau) = 0$. From the asymptotic behavior of $u(r)$ as $r \to \infty$ and Lemma 3 we have

$$\int_\tau^\infty r^{n-1}u^{p+1}(r)\left[\left(\frac{n}{p+1} - \frac{n-2}{2}\right)K(r) + \frac{1}{p+1}rK'(r)\right] dr$$
$$= -\frac{1}{p+1}\tau^n K(\tau)u(\tau)^{p+1} < 0.$$

This is contrary to condition (6).

Therefore $\tau = 0$. It is easy to see that $u(r)$ is not bounded as $r > 0$. Otherwise $u(r)$ is a weak solution of equation (1). By the elliptic regularity theory, $u(r)$ is a classical solution so that $u'(r) \to 0$ as $r \to 0$. Again by Lemma 3 we have

$$\int_0^\infty r^{n-1}u^{p+1}(r)\left[\left(\frac{n}{p+1} - \frac{n-2}{2}\right)K(r) + \frac{1}{p+1}rK'(r)\right] dr = 0, \tag{10}$$

a contradiction.

Now we use [GS, Theorem 3.4] to conclude that

$$r^{2/(p-1)}u(r) \to C_0 \quad \text{as} \quad r \to 0,$$

where $C_0 = [\frac{1}{K(0)} \cdot \frac{2(n-2)}{(p-1)^2}(p - \frac{n}{n-2})]^{1/(p-1)}$. □

Remark 1. We conjecture that under the conditions of Theorem 1, equation (1) has a singular solution with decay $r^{-2/(p-1)}$ at ∞.

Now we consider the existence of classical positive solutions of equation (9) with $K(r)$ satisfying condition (2). If $u(r)$ is a solution, by Lemmas 1 and 3 we have equality (10). Define a function $g(r)$ as follows:

$$g(r) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right) K(r) + \frac{1}{p+1} rK'(r). \tag{11}$$

The first necessary condition for existence of classical solutions is that $g(r)$ changes sign. From condition (2) we see $g(r)$ is positive for r small. Denote by r_1 the first zero of $g(r)$. Let

$$\begin{aligned} g^-(r) &= -\min\{0, g(r)\}, \\ g^+(r) &= \max\{0, g(r)\}. \end{aligned}$$

Since a classical solution of equation (9) satisfies the following initial value problem:

$$\begin{cases} u'' + \dfrac{n-1}{r}u' + K(r)u^p = 0, \\ u(0) = \alpha, \qquad u'(0) = 0, \end{cases} \tag{12}$$

we have

$$-r^{n-1}u'(r) = \int_0^r s^{n-1}K(s)u^p(s)\,ds, \tag{13}$$

and $u'(r) < 0$ for any $r > 0$. Writing (10) as

$$\int_0^{r_1} r^{n-1}u^{p+1}(r)g(r)\,dr + \int_{r_1}^{\infty} r^{n-1}u^{p+1}(r)g^+(r)\,dr = \int_{r_1}^{\infty} r^{n-1}u^{p+1}(r)g^-(r)\,dr,$$

we have

$$\begin{aligned} u^{p+1}(r_1)\int_0^{r_1} r^{n-1}g(r)\,dr &< \int_0^{r_1} r^{n-1}u^{p+1}(r)g(r)\,dr \\ &\le \int_{r_1}^{\infty} r^{n-1}u^{p+1}(r)g^-(r)\,dr \\ &< u^{p+1}(r_1)\int_{r_1}^{\infty} r^{n-1}g^-(r)\,dr. \end{aligned}$$

Hence we conclude that the second necessary condition is

$$\int_0^{r_1} r^{n-1}g(r)\,dr < \int_{r_1}^{\infty} r^{n-1}g^-(r)\,dr. \tag{14}$$

As a summary we have the following:

Lemma 4. *Under condition* (2) *and* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, *if equation* (9) *has classical positive solutions, then* $g(r)$ *defined in* (11) *satisfies the following*:

(i) $g(r)$ *changes sign*;

(ii) *inequality* (14) *holds, where* r_1 *is the first zero of* $g(r)$.

Remark 2. Since

$$\begin{aligned} g'(r) &= \left(\frac{n+1}{p+1} - \frac{n-2}{2}\right) K'(r) + \frac{r}{p+1} K''(r) \\ &= -\left(\frac{n-2}{2} - \frac{2}{p+1}\right) K'(r) + \frac{r}{p+1} \Delta K, \end{aligned}$$

it is easy to see that for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and $K(r)$ satisfying condition (2), $g(r)$ never changes sign in the following cases:

(i) $K' \geq 0$;

(ii) $K'(r) \leq 0$, $\Delta K \geq 0$;

(iii) $rK'(r) \to 0$ as $r \to \infty$ and $r\Delta K \leq \frac{1}{2}[p(n-2) + n - 6]K'(r)$;

so that equation (9) cannot have any classical solution.

Remark 3. The necessary conditions presented in Lemma 4 are not enough for existence of classical solutions. In the following we give an existence result for some kind of K's. It seems that for most of K's equation (1) does not have any classical solution. It is difficult but interesting to get existence results for more general functions K's.

From now on we assume that $K(r) = 1 + H(r)$, where $H(r)$ satisfies the following conditions:

$$(\mathrm{H}_1) \qquad \begin{cases} H \in C^2[0, \infty), \ H(r) \geq 0, \ H'(r) \leq 0, \ H'(0) = 0, \\ H(r) \to 0 \text{ and } rH'(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

Let $g(r) = (\frac{n}{p+1} - \frac{n-2}{2})(1 + H(r)) + \frac{1}{p+1} rH'(r)$. Keeping Lemma 4 in mind, we assume that $g(r)$ satisfies the following conditions:

$$(\mathrm{H}_2) \qquad \begin{cases} \text{There are } 0 < r_1 < r_2 \text{ such that } g(r) < 0 \text{ for } r_1 < r < r_2 \text{ and} \\ g(r) > 0 \text{ for } 0 \leq r < r_1 \text{ or } r > r_2, \text{ and} \\ \int_{B_{r_2}(0)} g(r)\, dx \leq 0. \end{cases}$$

From (H_2) we can construct a nontrivial radial function $\varphi \in C_0^2(B_{r_2}(0))$, $\varphi \geq 0$, such that

$$\int_{B_{r_2}(0)} \varphi(r) g(r)\, dx \leq 0. \tag{15}$$

Let $H_r(B_R)$ be the subspace of $H_0^1(B_R)$ consisting of radial functions, where $B_R = B_R(0)$. Define $E(R)$ as a subset of $H_r(B_R)$ consisting of functions which satisfy the following conditions:

$$\begin{aligned} &\int_{B_R} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1 + H)|u|^{p+1}\, dx = 1 \\ &\int_{B_R} \frac{1}{p+1} rH'(r)|u|^{p+1}\, dx = -1. \end{aligned} \tag{16}$$

Lemma 5. *Under conditions* (H_1) *and* (H_2), $E(R)$ *is a nonempty* C^1 *manifold for each* $R \geq r_2$.

Proof. Take φ as in (15). For each $\rho > 0$, $\varphi(\rho^{-1}r) \in H_r(B_{\rho r_2})$. Set $J(\rho)$ as follows:

$$J(\rho) = \int_{B_{\rho r_2}} \varphi(\rho^{-1}r)^{p+1} g(r)\, dx = \rho^n \int_{B_{r_2}} \varphi(r)^{p+1} g(\rho r)\, dx.$$

By (H_2), $J(1) \leq 0$ and $J(\rho) > 0$ for ρ small. Therefore $J(\rho_0) = 0$ for some $0 < \rho_0 \leq 1$, that is,

$$\int_{B_{\rho_0 r_2}} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1+H)\varphi(\rho_0^{-1}r)^{p+1}\, dx$$
$$= -\int_{B_{\rho_0 r_2}} \frac{1}{p+1} rH'(r)\varphi(\rho_0^{-1}r)^{p+1}\, dx.$$

Let $v(r) = C\varphi(\rho_0^{-1}r)$ for some suitable $C > 0$. Then $v \in E(\rho_0 r_2)$. Obviously $E(\rho_0 r_2) \subset E(R)$ for each $R \geq r_2$. Hence $E(R)$ is nonempty for every $R \geq r_2$.

Define two functionals f_1, f_2 as follows:

$$f_1(u) = \int_{B_R} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1+H)|u|^{p+1}\, dx - 1,$$
$$f_2(u) = \int_{B_R} \frac{1}{p+1} rH'(r)|u|^{p+1}\, dx + 1.$$

Then f_1, f_2 are differentiable functionals on $H_r(B_R)$ and

$$E(R) = \{u \in H_r(B_R) : f_1(u) = f_2(u) = 0\}.$$

If $f'(u) = \lambda f_2'(u)$ for some $u \in E(R)$ and $\lambda \in R^1$, for any $v \in H_r(B_R)$ then we have

$$\int_{B_R} \left[\left(\frac{n}{p+1} - \frac{n-2}{2}\right)(p+1)(1+H) - \lambda rH'(r)\right] |u|^{p-1}uv\, dx = 0.$$

Taking $v = u$ we have

$$\int_{B_R} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1+H)|u|^{p+1}\, dx = \lambda \int_{B_R} \frac{1}{p+1} rH'(r)|u|^{p+1}\, dx.$$

Hence $\lambda = -1$ and

$$\int_{B_R} g(r)|u|^{p-1}uv\, dx = 0, \qquad \text{for any } v \in H_r(B_R),$$

which implies $g(r)|u|^{p-1}u = 0$, a.e. For $r \neq r_1$, $r \neq r_2$, $g(r) \neq 0$, so that $u = 0$, a.e., which is contrary to (16). It follows that the mapping F defined by

$$F \colon H_r(B_R) \to R^2, \qquad F(u) = (f_1(u), f_2(u)),$$

is transversal to $\{\theta\}$, where θ is the zero function in $H_r(B_R)$. Therefore $E(R) = F^{-1}(0)$ is a C^1 submanifold of $H_r(B_R)$ and has the natural Finsler structure (see [C, pp. 32, 85]). □

Now we are able to prove our main theorem.

Theorem 2. *In addition to* (H_1) *and* (H_2), *assume* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ *and*

$$r^2 \Delta H \geq C_1 r H'(r) - C_2(1 + H(r)), \tag{17}$$

where

$$\begin{aligned} &C_1 < \tfrac{1}{2}[p(n-2) + n - 6], \\ &C_2 < \tfrac{1}{4}[n + 2 - p(n-2)] \cdot [p(n-2) + n - 6 - 2C_1]. \end{aligned} \tag{18}$$

Then equation (9) *has a classical positive solution for* $K(r) = 1 + H(r)$.

Proof. For each $R \geq r_2$, consider the following minimizing problem:

$$A(R) = \inf_{u \in E(R)} \int_{B_R} |\nabla u|^2 \, dx. \tag{19}$$

It is obvious that $A(R) > 0$ and there is $u_R \in E(R)$ such that

$$\int_{B_R} |\nabla u_R|^2 \, dx = A(R). \tag{20}$$

Replacing $u_R(r)$ by $|u_R(r)|$ if necessary, $u_R(r) \geq 0$ for $0 \leq r < R$. By Lagrangian method there are constants λ_R, δ_R such that u_R satisfies the following Euler equation:

$$\begin{cases} -\Delta u_R = [\lambda_R(1 + H(r)) + \delta_R r H'(r)] u_R^p, & \text{in } B_R \\ u_{R|\partial B_R} = 0. \end{cases} \tag{21}$$

From the elliptic regularity theory $u_R \in C^2(B_R)$. By using Lemma 3 and (16), a direct calculation shows

$$\delta_R \left[(p(n-2) + n - 6) + \frac{1}{p+1} \int_{B_R} (r^2 \Delta H) u_R^{p+1} \, dx \right] = \frac{1}{2} \omega_n R^n |u_R'(R)|^2. \tag{22}$$

From (17) we have

$$\begin{aligned} &\frac{1}{p+1} \int_{B_R} (r^2 \Delta H) u_R^{p+1} \, dx \\ &\quad \geq \frac{1}{p+1} \int_{B_R} [C_1 r H'(r) - C_2(1 + H(r))] u_R^{p+1} \, dx \\ &\quad = -C_1 - \frac{C_2}{p+1} \left(\frac{n}{p+1} - \frac{n-2}{2} \right)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}[p(n-2) + n - 6] + \frac{1}{p+1} \int_{B_R} (r^2 \Delta H) u_R^{p+1} \, dx \\ &\quad \geq \frac{1}{2}[p(n-2) + n - 6 - 2C_1] - \frac{C_2}{p+1} \left(\frac{n}{p+1} - \frac{n-2}{2} \right)^{-1} > 0. \end{aligned} \tag{23}$$

Equations (22) and (23) imply $\delta_R > 0$.

Choosing $R = m$ and letting $m \to \infty$, we have a sequence $\{u_m, \lambda_m, \delta_m\}$. Since $E(m) \subset E(m+1)$, $\{A(m)\}$ is a decreasing sequence, and

$$
\begin{aligned}
A(m) &= \int_{B_m} |\nabla u_m|^2 \\
&= \int_{B_m} [\lambda_m(1 + H(r)) + \delta_m r H'(r)] u_m^{p+1}\, dx \qquad (24)\\
&= \lambda_m \left(\frac{n}{p+1} - \frac{n-2}{2}\right)^{-1} - \delta_m(p+1).
\end{aligned}
$$

$\delta_m > 0$ implies $\lambda_m > 0$.

Assume $\lambda_m \to +\infty$. From (24) we get $\delta_m \to +\infty$ and

$$
\frac{\lambda_m}{\delta_m} = \left[(p+1) + \frac{A(m)}{\delta_m}\right] \cdot \left(\frac{n}{p+1} - \frac{n-2}{2}\right) \to (p+1)\left(\frac{n}{p+1} - \frac{n-2}{2}\right). \qquad (25)
$$

Set $u_m(x) = u_m(|x|)$ and assume $u_m(x)$ reaches the maximum value α_m at x_m. Set

$$
v_m(y) = \frac{1}{\alpha_m} u_m(x_m + \beta_m y),
$$

where $\beta_m = (\lambda_m \alpha_m^{p-1})^{-1/2}$. Then v_m satisfies the following equation:

$$
\Delta v_m + \left[1 + H(|x_m + \beta_m y|) + \frac{\delta_m}{\lambda_m} B(|x_m + \beta_m y|)\right] v_m^p = 0, \qquad (26)
$$

where $B(r) = rH'(r)$.

Now we claim that $\beta_m \to 0$. Since $\lambda_m \to \infty$, it suffices to check that $\alpha_m \geq C$ for some positive constant C.

For $R > r_2$, denote $T(R) = \max_{r \geq R} r|K'(r)| = \max_{r \geq R} r|H'(r)|$. From (H_1) we have $T(R) \to 0$ as $R \to \infty$. From (16) we have

$$
\begin{aligned}
p + 1 &= \int_{B_m} r|H'(r)| u_m^{p+1}\, dx \\
&= \int_{B_R} r|H'(r)| u_m^{p+1}\, dx + \int_{R \leq |x| \leq m} r|H'(r)| u_m^{p+1}\, dx \\
&\leq \int_{B_R} r|H'(r)| u_m^{p+1}\, dx + T(R) \int_{B_m} u_m^{p+1}\, dx \\
&\leq \int_{B_R} r|H'(r)| u_m^{p+1}\, dx + T(R)(1 + H(0))^{-1} \int_{B_m} (1 + H) u_m^{p+1}\, dx \\
&= \int_{B_R} r|H'(r)| u_m^{p+1}\, dx + T(R)(1 + H(0))^{-1} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)^{-1}.
\end{aligned}
$$

Choose R_0 large enough such that

$$
T(R_0) < \frac{1}{2}(p+1)(1 + H(0)) \cdot \left(\frac{n}{p+1} - \frac{n-2}{2}\right).
$$

For this R_0 we have

$$\frac{1}{2}(p+1) \le \int_{B_{R_0}} r|H'(r)|u_m^{p+1}\,dx \le \alpha_m^{p+1} \int_{B_{R_0}} r|H'(r)|\,dx,$$

which implies that our claim is valid.

Since $\{v_m\}$ is uniformly bounded and (25) holds, by the elliptic estimates we conclude that $\{v_m\}$ is precompact in $C^2_{\text{loc}}(R^n)$. Hence there is a subsequence, still denoted by $\{v_m\}$, such that

$$v_m \to v_0 \text{ in } C^2(B_R) \quad \text{for any } R > 0.$$

Case 1. $|x_m| \to \infty$. By (H_1) and the fact $\beta_m \to 0$, taking the limit as $m \to \infty$ in (26), we see v_0 is a solution of the equation

$$\Delta v + v^p = 0, \tag{27}$$

and $v_0(0) = 1$. Hence v_0 is a classical positive solution of equation (27), a contradiction to Lemma 2.

Case 2. $\{x_m\}$ has a bounded subsequence. Without loss of generality, assume $x_m \to x_0$. Taking the limit as $m \to \infty$ in (26), we see v_0 is a solution of the equation

$$\Delta v + Cv^p = 0, \tag{28}$$

where

$$\begin{aligned} C &= 1 + H(r_0) + \frac{1}{p+1}\left(\frac{n}{p+1} - \frac{n-2}{2}\right) r_0 H'(r_0) \\ &= \left(\frac{n}{p+1} - \frac{n-2}{2}\right) g(r_0), \end{aligned}$$

and $r_0 = |x_0|$. We know that equation (28) does not have any classical solution when $C \ne 0$ (when $C > 0$ this is a direct consequence, when $C < 0$ see [N2, Theorem 1.15]). Therefore $g(r_0) = 0$, that is, $r_0 = r_1$ or $r_0 = r_2$.

Let $\rho_m = |x_m|$; then $\rho_m \to r_0$. Since u_m satisfies

$$u_m'' + \frac{n-1}{r}u_m' + [\lambda_m(1 + H(r)) + \delta_m r H'(r)]u_m^p = 0,$$

by using (25), we rewrite the above equation

$$u_m'' + \frac{n-1}{r}u_m' + \delta_m(p+1)[g(r) + D_m^{(r)}]u_m^p = 0, \tag{29}$$

where

$$D_m(r) = \frac{1}{p+1}\left(\frac{n}{p+1} - \frac{n-2}{2}\right)\frac{A(m)}{\delta_m}(1 + H(r)).$$

Noting $\lambda_m > 0$, we know that u_m has a local maximum at $r = 0$, therefore u_m has a local minimum at $\tau_m \in (0, \rho_m)$. From (29) and the fact $u_m'(\tau_m) = 0$, $u_m''(\tau_m) > 0$, we have $g(\tau_m) + D_m(\tau_m) < 0$, that is, $r_1 < \tau_m < r_2$. From the same reason we have $g(\rho_m) + D_m(\rho_m) > 0$. If $r_0 = r_1$, since $D(\rho_m) \to 0$ as $m \to \infty$, for m large we have

$$r_1 < \tau_m < \rho_m < \frac{1}{2}(r_1 + r_2) \quad \text{and} \quad g(\rho_m) + D_m(\rho_m) < 0,$$

a contradiction.

Therefore $r_0 = r_2$ and $\rho_m \to r_2$ as $m \to \infty$. Now it is clear that $g(r) + D_m(r) > 0$ for $r > \rho_m$ and $u_m(r)$ is decreasing in $\rho_m < r < m$.

Fix $0 < \varepsilon < 1$, take $b_\varepsilon > 0$ such that

$$(p+1)(g(r) + D_m(r)) \geq 2b_\varepsilon$$

for $r \geq (1-\varepsilon)^{-1/n} r_2$. Let $z_m = (1-\varepsilon)^{-1/n}\rho_m$. Since $u_m'(\rho_m) = 0$, from (29) we know that for $z_m \leq r \leq m$,

$$\begin{aligned} -r^{n-1}u_m'(r) &= \int_{\rho_m}^r s^{n-1}\delta_m(p+1)[g(s) + D_m(s)]u_m^p(s)\,ds \\ &\geq b_\varepsilon \delta_m u_m^p(r) \cdot \tfrac{1}{n}(r^n - \rho_m^n) \geq \tfrac{\varepsilon}{n} b_\varepsilon \delta_m u_m^p(r) r^n, \end{aligned}$$

and

$$-\frac{u_m'}{u_m^p} \geq \frac{\varepsilon}{n} b_\varepsilon \delta_m r. \tag{30}$$

Integrating (30) from z_m to r gives

$$u_m(r) \leq C_\varepsilon \delta_m^{-1/(p-1)} r^{-2/(p-1)}, \quad \text{for } z_m \leq r \leq m, \tag{31}$$

where

$$C_\varepsilon = \left[\frac{p-1}{2n}\varepsilon b_\varepsilon\right]^{-1/(p-1)}.$$

Since $\sup_m \int_{B_m} |\nabla u_m|^2\,dx < \infty$, we can choose some $r_3 > (1-\varepsilon)^{-1/n} r_2$ such that

$$\sup_m r_3^{n-1}|u_m'(r_3)| \leq C,$$

therefore, for $r_3 < r < m$ we have

$$\begin{aligned} -r^{n-1}u_m'(r) &= -r_3^{n-1}u_m'(r_3) + \int_{r_3}^r s^{n-1}\delta_m(p+1)[g(s) + D_m(s)]u_m^p(s)\,ds \\ &\leq C + M\delta_m \int_{r_3}^r s^{n-1}[\delta_m^{-1/(p-1)} s^{-2/(p-1)}]^p\,ds \\ &< C + M\delta_m^{-1/(p-1)}\left(n - \frac{2p}{p-1}\right)^{-1} r^{n-2p/(p-1)}, \end{aligned}$$

where $M = \sup_{m\geq 1,\ r \geq r_3}(p+1)(g(r) + D_m(r))$. For m large we have

$$m^{n-1}|u_m'(m)| \geq M m^{n-2p/(p-1)},$$

that is,

$$|u_m'(m)| \leq M m^{-(1+2/(p-1))}.$$

Now taking $R = m$ in (22), we have

$$\begin{aligned} &\delta_m\left[\frac{1}{2}(p(n-2) + n - 6) + \frac{1}{p+1}\int_{B_m}(r^2\Delta H)u_m^{p+1}\,dx\right] \\ &\quad = \tfrac{1}{2}\omega_n m^n |u_m'(m)|^2 \\ &\quad \leq \tfrac{1}{2}\omega_n M^2 m^{n-2-4/(p-1)} \to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{32}$$

From (23) the left-hand term tends to ∞ as $m \to \infty$. Again we reach a contradiction.

Hence we conclude that $\{\lambda_m\}$ is bounded, and from (24), $\{\delta_m\}$ is bounded.

Without loss of generality, we assume

$$\begin{aligned} &\lambda_m \to \lambda_0, \qquad \delta_m \to \delta_0, \\ &u_m \to u_0 \text{ weakly in } L^{p+1}(R^n), \\ &\nabla u_m \to \nabla u_0 \text{ weakly in } L^2(R^n), \text{ and} \\ &u_m \to u_0 \text{ strongly in } L^{p+1}_{\text{loc}}(R^n). \end{aligned}$$

It is easy to see that u_0 is a nonnegative radial function. By Fatou's Lemma we have

$$\int_{R_n} \left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1+H(r))u_0^{p+1}\,dx \le 1, \tag{33}$$

$$\int_{R^n} \frac{1}{p+1}(-rH'(r))u_0^{p+1}\,dx \le 1. \tag{34}$$

Since

$$\begin{aligned} &\int_{R\le|x|\le m} (-rH'(r))u_m^{p+1}\,dx \\ &\quad \le T(R)\int_{R\le|x|\le m} u_m^{p+1}\,dx \\ &\quad \le T(R)\left(\frac{n}{p+1} - \frac{n-2}{2}\right)(1+H(0))^{-1} \to 0 \quad \text{as } R\to\infty, \end{aligned}$$

where $T(R) = \sup_{r\ge R}|rH'(r)|$, for any $0<\varepsilon<1$ we can choose R_ε large such that

$$\int_{B_{R_\varepsilon}} \frac{1}{p+1}(-rH'(r))u_m^{p+1}\,dx \ge 1-\varepsilon.$$

Since $u_m \to u_0$ strongly in $L^{p+1}(B_{R_\varepsilon})$, we have

$$\int_{B_{R_\varepsilon}} \frac{1}{p+1}(-rH'(r))u_0^{p+1}\,dx \ge 1-\varepsilon,$$

which implies

$$\int_{R^n} \frac{1}{p+1}(-rH'(r))u_0^{p+1}\,dx \ge 1. \tag{35}$$

Equations (34) and (35) together mean

$$\int_{R^n} \frac{1}{p+1}(-rH'(r))u_0^{p+1}\,dx = 1. \tag{36}$$

Assuming $\lambda_m \to 0$, (24) gives $\delta_m \to 0$, therefore,

$$\int_{R^n} |\nabla u_0|^2\,dx \le \liminf_{m\to\infty} \int_{R^n} |\nabla u_m|^2 = 0$$

which is contrary to (36). Hence $\lambda_0 > 0$. By the same argument we can prove that (32) is valid for $\delta_m \to \delta_0$. Using (23) again, we conclude that $\delta_m \to 0$. Therefore u_0 is a nonnegative radial solution of the following equation in R^n:

$$\Delta u_0 + \lambda_0(1+H(r))u_0^p = 0.$$

By maximum principle we know $u_0(r) > 0$. From the elliptic regularity theory u_0 is a classical solution. Now let $u(r) = \lambda_0^{-1/(p-1)} u_0(r)$, which is a classical solution of equation (9) for $K(r) = 1 + H(r)$. □

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Department of Applied Mathematics, Zhejiang University, Hangzhou, People's Republic of China

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455