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ON REAL INVARIANT SUBSPACES OF BOUNDED OPERATORS WITH COMPACT IMAGINARY PART

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ABSTRACT. In this paper we show the existence of real invariant subspaces for compact perturbations of selfadjoint operators.

1. INTRODUCTION

One of the most difficult questions in the theory of invariant subspaces is the problem of the existence of invariant subspaces for a compact perturbation of a selfadjoint operator. Livsič solved this problem for nuclear perturbations, Sahnovič for Hilbert-Schmidt perturbations, and Gohberg and Kreĭn, Macaev, and Schwartz for the perturbation being in the Schatten's class $\mathscr{C}^p$. Macaev proved the spectral separation theorem for a perturbation in the class $\mathscr{S}_w$, and moreover, if the operator does not belong to the class $\mathscr{S}_w$ then its spectrum cannot be separated in general. (Recall that the operator K belongs to the class $\mathscr{S}_w$ if its s-numbers satisfy $\sum_{j=1}^{\infty} s_j(K)/(2j-1) < \infty$.) However, there is still hope to prove the existence of invariant subspaces for every compact perturbation of a selfadjoint operator. In this paper we do so for real subspaces.

2. THE RESULT

Let $\mathscr{H}$ be a complex Hilbert space. A subset of $\mathscr{H}$ is called a real subspace if it is closed under vector addition and multiplication by real scalars. Let A be a bounded linear operator on $\mathscr{H}$, and suppose that $A = R + K$, where R is a selfadjoint operator and K is a compact operator. In this paper we prove the following result.

Theorem 1. *Let A be a bounded linear operator on $\mathscr{H}$, and suppose that $A = R + K$, where R is selfadjoint and K is compact. Then A has a nontrivial closed real invariant subspace.*

In order to prove the theorem we need the following well-known fact from the theory of invariant subspaces. Recall that an operator A on the space $\mathscr{H}$ is said to be quasi-triangular if there exists a sequence $\{Q_n\}_{n=1}^{\infty}$ of finite rank

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orthogonal projections strongly converging to I such that

$$\lim_{n\to\infty} \|(I - Q_n)AQ_n\| = 0.$$

Lemma 1. *If A is quasi-triangular then there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections that converges weakly to a selfadjoint operator S other than 0 or I with the property that*

$$\lim_{n\to\infty} \|(I - P_n)AP_n\| = 0.$$

(For the proof of the lemma see, e.g., [1, p. 85].)

Let us make several remarks.

(1) The operator A from the theorem is obviously quasi-triangular. Indeed, the spectral theorem implies that a selfadjoint operator is quasi-triangular, and a compact perturbation preserves this property.

(2) Any positive power A^ℓ can be represented as $A^\ell = R_\ell + K_\ell$, where R_ℓ is selfadjoint and K_ℓ is compact.

(3) For any positive integer ℓ the following assertion holds:

$$\lim_{n\to\infty} \|(I - P_n)A^\ell P_n\| = 0.$$

Note that this formula means that an almost invariant subspace of A must be almost invariant for any positive power of A. Indeed, the case $\ell = 1$ is covered by lemma, and for $\ell > 1$ we use induction,

$$\begin{aligned}\|(I - P_n)A^\ell P_n\| &= \|(I - P_n)A^{\ell-1}(I - P_n + P_n)AP_n\| \\ &\le \|(I - P_n)A^{\ell-1}P_nAP_n\| + \|(I - P_n)A^{l-1}(I - P_n)AP_n\| \\ &\le C_1\|(I - P_n)A^{\ell-1}P_n\| + C_2\|(I - P_n)AP_n\|.\end{aligned}$$

Returning to the proof of the theorem. We have

$$\begin{aligned}A^\ell S - SA^\ell S &= (weak) - \lim_{n\to\infty}(A^\ell P_n - SA^\ell S) \\ &= (weak) - \lim_{n\to\infty}(P_nR_\ell P_n - SR_\ell S) \\ &\quad + (weak) - \lim_{n\to\infty}(P_nK_\ell P_n - SK_\ell S).\end{aligned}$$

The second term of the last sum converges to zero thanks to the compactness of the operator K_ℓ, and the first term converges to a selfadjoint operator T_ℓ. Therefore we have $(I - S)A^\ell S = T_\ell$.

Let x be a vector in $\mathscr{H}$ such that both $Sx \neq 0$ and $(I - S)x \neq 0$. Then for any positive integer ℓ we have

$$\operatorname{Im}\langle A^\ell Sx, (I - S)x\rangle = \operatorname{Im}\langle T_\ell x, x\rangle = 0.$$

Consider the real closed subspace L of $\mathscr{H}$ spanned by the vectors $\{A^\ell Sx\}$. Obviously this subspace does not contain the vector $i(I - S)x$ and thus L is a nontrivial invariant subspace of the operator A. Thus the proof of the theorem is complete.

Note that the subspaces $L \cap iL$ and $\overline{\operatorname{span}}(L, iL)$ are complex invariant subspaces of the operator A. Assuming that the operator A does not have invariant subspaces, we have $L \cap iL = \{0\}$ and $\overline{\operatorname{span}}(L, iL) = \mathscr{H}$.

If, in addition, L and iL were orthogonal then the operator A would have a real form.

Therefore the lack of existence of invariant subspaces for A gives the reward of a real form for A.

The preceding argument leads to the following question: Does there exist real invariant subspaces for an arbitrary bounded operator on a complex Hilbert or Banach space?

References

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