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A GENERIC H-INVARIANT IN A MULTIPLICITY-FREE G-ACTION

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ABSTRACT. For a prehomogeneous action of a reductive group G on a vector space V, we construct a formal power series L that is shown to have a nonzero projection to every G-isotypic component of $P(V)$. When (G, V) is multiplicity-free, these "Fourier components" of the function L provide all the H-invariants in $P(V)$ for some spherical subgroup H of G. Three interesting examples are presented.

I. MAIN RESULT

Let G be a connected reductive complex algebraic group, $G \supseteq \mathbb{C}^\times$. Suppose that G acts on a complex vector space V *prehomogeneously*, i.e., G has a Zariski open (hence dense) orbit in V. From [Se], G acts on V^* prehomogeneously. Fix $0 \neq \phi$ in the open G-orbit in V^*.

Consider $L = e^\phi = \sum_{n \geq 0} \phi^n/n!$ as a formal power series on V.

Let

$$P(V) = \sum_{\sigma \in \widehat{G}} P(V)_\sigma$$

be the G-isotypic decomposition of $P(V)$, the algebra of polynomial functions on V.

Theorem 1. *L has a nonzero projection to all $P(V)_\sigma$ such that $P(V)_\sigma \neq 0$.*

Let H be the stabilizer in G of ϕ. By definition, ϕ is an H-invariant, as is the projection of $L = e^\phi$ to $P(V)_\sigma$. Let

$$L = \sum_{\sigma \in \widehat{G}} L_\sigma$$

be the G-isotypic decomposition of L, thus we have $0 \neq L_\sigma \in P(V)_\sigma \cap P(V)^H$, if $P(V)_\sigma \neq 0$.

Now suppose the action of G on V is *multiplicity-free*, namely, the natural action of G on $P(V)$ is multiplicity-free as a representation of G. From [Se, VK], G acts on V (V^*, resp.) prehomogeneously. Let ϕ, H, L be defined as in above. In this case, since $H \subseteq G$ is spherical (see [HU] for example), i.e., $\dim \pi^H \leq 1$ for any irreducible finite-dimensional representation of G, and

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since (G, V) is multiplicity-free, we see that L_σ is the unique (up to scalar) H-invariant in $P(V)_\sigma$, if $P(V)_\sigma \neq 0$. Thus we have

Corollary 2. *$P(V)^H =$ linear span of $\{L_\sigma\}_{\sigma\in\widehat{G}}$.*

In other words, the "Fourier components" of the simple-minded function L provide all the H-invariants.

Proof of Theorem 1. We have

$$P(V^*) \cong S(V) \cong D(V), \tag{1}$$

where $S(V)$ is the symmetric algebra on V and $D(V)$ is the algebra of constant coefficient differential operators on $P(V)$. The first isomorphism is the obvious one, and the second isomorphism is induced by the following:

$$(v_0 \cdot f)(v) = \frac{d}{dt} f(v + tv_0)|_{t=0}, \qquad v, v_0 \in V.$$

For $q \in P(V^*)$, we denote by $\partial_q \in D(V)$ the corresponding constant coefficient differential operator given by (1).

We have the natural pairing between $P(V)$ and $P(V^*)$ defined by

$$(f, q) = (\partial_q f)(0).$$

This pairing is nondegenerate on each homogeneous component $P^n(V)$ and $P^n(V^*)$, and therefore induces a natural identification of $P^n(V^*)$ with $P^n(V)^*$, the contragredient of $P^n(V)$. But since $G \supseteq C^\times$, each G-isotypic component $P(V)_\sigma$ is contained in $P^n(V)$ for some n. Thus dual to the given G-isotypic decomposition of $P(V)$, we have the G-isotypic decomposition of $P(V^*)$

$$P(V^*) \cong \sum_{\sigma\in\widehat{G}} P(V)_\sigma^*.$$

Since $(v_0 \cdot e^\phi)(v) = \frac{d}{dt} e^{\phi(v+tv_0)}|_{t=0} = \phi(v_0)e^{\phi(v)}$ for $v, v_0 \in V$, we obtain

$$(e^\phi, q) = (\partial_q e^\phi)(0) = q(\phi), \qquad q \in P(V^*).$$

Thus the projection of e^ϕ to $P(V)_\sigma$ is zero if and only if $q(\phi) = 0$ for all $q \in P(V)_\sigma^*$. But since $P(V)_\sigma^*$ is a G-isotypic component of $P(V^*)$, all elements of it vanish at ϕ if and only if they all vanish at $g \cdot \phi$ for any $g \in G$, that is, if and only if they vanish on the whole G-orbit $G \cdot \phi$. But since this orbit is dense in V^*, this implies the elements of $P(V)_\sigma^*$ are identically zero, a contradiction.

II. A relation between $PD(V)^G$ and $P(V)^H$

Let $PD(V)$ be the algebra of polynomial coefficient differential operators on $P(V)$. The following proposition was originally used to prove Theorem 1 in the case of multiplicity-free actions. We give it here because of its independent interest.

Proposition 3. *If (G, V) is multiplicity-free, then $PD(V)^G L = P(V)^H L$.*

Remark. It is quite trivial that $PD(V)^G L \subseteq P(V)^H L$. The interest in the proposition is that you get them all, in fact, in a clean and nonredundant way (see the following proof).

Proof. Since the G-invariant set in $V \oplus V^*$ swept out by $V + \phi$ is $G(V+\phi) = V + G\phi$, which is dense in $V \oplus V^*$, we have an injective homomorphism

$$r \colon P(V \oplus V^*)^G \mapsto P(V)^H$$
$$(rQ)(v) = Q(v, \phi), \qquad v \in V.$$

Since the action of G on V is multiplicity-free, r is an algebra isomorphism. See [HU]. We have canonical isomorphisms

$$P(V \oplus V^*) \cong P(V) \otimes P(V^*) \cong P(V) \otimes S(V) \cong P(V) \otimes D(V) \cong PD(V). \tag{2}$$

Let $Q \in P(V \oplus V^*)$, and let $D_Q \in PD(V)$ be the polynomial differential operator corresponding to Q from (2).

Then again since $(v_0 \cdot e^{\phi})(v) = \frac{d}{dt} e^{\phi(v+tv_0)}|_{t=0} = \phi(v_0)e^{\phi(v)}$, $v, v_0 \in V$, we have

$$(D_Q L)(v) = (D_Q e^{\phi})(v) = Q(v, \phi)e^{\phi} = (rQ)(v)L. \tag{3}$$

Since all the isomorphisms in (2) are G-isomorphisms, we have

$$PD(V)^G \cong P(V \oplus V^*)^G \underset{\cong}{\overset{r}{\mapsto}} P(V)^H.$$

Combining the above with (3), the proposition follows.

III. A refined version of Theorem 1

If (G, V) is multiplicity-free, Theorem 1 can be strengthened in the following way.

From [VK], a Borel subgroup B of G has an open orbit on V^*. This means that B has an open dense orbit in $G \cdot \phi \cong G/H$, which is the same as saying that there is an open dense (B, H) double coset in G. By replacing the Borel subgroup B with a conjugate of B, we can and do assume our Borel subgroup is chosen so that BH is open dense in G. By looking at the tangent space at the identity of G, we conclude

$$\mathfrak{g} = \mathfrak{b} + \mathfrak{h},$$

where $\mathfrak{g}$ is the Lie algebra of G, etc. (This can also be dug out of [VK].)

Let $A \subseteq B$ be the Cartan subgroup. With respect to the above choice of B, there is a unique (up to scalar) highest weight vector in $P(V)_\sigma$, denoted by v_σ.

Let

$$L_\sigma = \lambda_\sigma v_\sigma + N_\sigma,$$

where N_σ is a sum of A-weight vectors in $P(V)_\sigma$ with weights strictly less than that of v_σ.

Theorem 1′. *$\lambda_\sigma \neq 0$ for any σ such that $P(V)_\sigma \neq 0$.*

Proof. By Theorem 1, $L_\sigma \neq 0$. Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, etc. Then by the irreducibility of $P(V)_\sigma$, we have

$$\mathscr{U}(\mathfrak{g})L_\sigma = P(V)_\sigma.$$

Since $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$, we have $\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{b})\mathscr{U}(\mathfrak{h})$ by the Poincare-Birkhoff-Witt theorem.

Now suppose $\lambda_\sigma = 0$. Since L_σ is H-invariant, we have

$$\mathscr{U}(\mathfrak{g})L_\sigma = \mathscr{U}(\mathfrak{b})\mathscr{U}(\mathfrak{h})L_\sigma \subseteq \mathscr{U}(\mathfrak{b})L_\sigma = \mathscr{U}(\mathfrak{b})N_\sigma .$$

Since each vector in $\mathscr{U}(\mathfrak{b})N_\sigma$ is a sum of A-weight vectors with weights strictly less than that of v_σ, $\mathscr{U}(\mathfrak{g})L_\sigma \subsetneq P(V)_\sigma$, a contradiction.

IV. Three examples

We consider the following three multiplicity-free actions of group G on the vector space V [Sh, HU].

(A) $G = Gl_k(\mathbb{C})$, $V = S_k$, the space of $k \times k$ complex symmetric matrices.

$$g \cdot p = (g^t)^{-1} p g^{-1}, \qquad g \in G,\ p = (p_{ij})_{k\times k} \in V .$$

(B) $G = Gl_k(\mathbb{C}) \times Gl_k(\mathbb{C})$, $V = M_k$, the space of $k \times k$ complex matrices.

$$(g_1, g_2) \cdot p = (g_1^t)^{-1} p g_2^t, \qquad g = (g_1, g_2) \in G,\ p = (p_{ij})_{k\times k} \in V .$$

(C) $G = Gl_{2k}(\mathbb{C})$, $V = \Lambda_k$, the space of $2k \times 2k$ complex skew-symmetric matrices.

$$g \cdot p = (g^t)^{-1} p g^{-1}, \qquad g \in G,\ p = (p_{ij})_{2k\times 2k} \in V .$$

We can identify V^* with V via the nondegenerate quadratic form

$$(p_1, p_2) = tr(p_1^t p_2), \qquad p_1, p_2 \in V .$$

Under this identification, the open G-orbit in V (V^*, resp.) consists of all nondegenerate matrices in V.

Let $\phi \in V^* \subseteq P(V)$ be the following linear functional on V:

$$\phi(p) = p_{11} + p_{22} + \cdots + p_{kk}, \qquad p \in V = S_k ;$$
$$\phi(p) = p_{11} + p_{22} + \cdots + p_{kk}, \qquad p \in V = M_k ;$$
$$\phi(p) = 2(p_{1,2} + p_{3,4} + \cdots + p_{2k-1,2k}), \qquad p \in V = \Lambda_k .$$

It corresponds to the following element in V:

$$I_k = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \qquad I_k,\ J_k = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix},$$

respectively. H, the stabilizer of ϕ in G, is isomorphic to $O_k(\mathbb{C})$, $Gl_k(\mathbb{C})$, $Sp_{2k}(\mathbb{C})$.

Let us give the explicit descriptions of $P(V)_\sigma$.

(A) $G = Gl_k(\mathbb{C})$, $V = S_k$. As usual, we parametrize the irreducible representations of Gl_k by their highest weights with respect to the upper triangular Borel subgroup or the corresponding Young diagrams. Thus, we use D to denote a decreasing sequence of nonnegative integers,

$$D = (a_1, a_2, \ldots, a_k), \qquad a_i \geq a_{i+1},\ a_i \in \mathbb{Z}^+,$$

and ρ_k^D is the irreducible representation of Gl_k with highest weight parametrized by D with respect to the standard coordinates on the diagonal torus. Let $|D| = a_1 + a_2 + \cdots + a_k$ if $D = (a_1, a_2, \ldots, a_k)$.

It is well known (see [H1, Sh] for example) that

$$P(S_k) \cong \sum_{D \text{ has even rows}} \rho_k^D ,$$

here we say D has even rows if D has the form $D = (2a_1, 2a_2, \ldots, 2a_k)$.

In fact, one has explicit descriptions of all the highest weight vectors, they are

$$d_1^{a_1-a_2} d_2^{a_2-a_3} \cdots d_{k-1}^{a_{k-1}-a_k} d_k^{a_k} ,$$

where

$$d_t = \det \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1t} \\ p_{21} & p_{22} & \cdots & p_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ p_{t1} & p_{t2} & \cdots & p_{tt} \end{pmatrix} ,$$

$a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are integers.

The corresponding irreducible representation ρ_k^D of GL_k has the highest weight $(2a_1, 2a_2, \ldots, 2a_k)$. One observes that all the polynomial functions in $P(V)_{\rho_k^D}$ are homogeneous of total degree $a_1 + a_2 + \cdots + a_k = \frac{1}{2}|D|$.

Thus in this case, our main result can be rephrased as

Theorem 1(A). *For any integer $n \geq 0$, the polynomial function*

$$(p_{11} + p_{22} + \cdots + p_{kk})^n$$

has a nonzero projection to all ρ_k^D with D even and $|D| = 2n$. Moreover, these projections span linearly the space of $O_k(\mathbb{C})$-invariants in $P^n(S_k)$.

(B) $G = Gl_k(\mathbb{C}) \times Gl_k(\mathbb{C})$, $V = M_k$. It is well known (see [Zhe]) that

$$P(M_k) \cong \sum \rho_k^D \otimes (\rho_k^D)^* .$$

In fact, the simultaneous highest and lowest weight vectors are of the following form:

$$d_1^{a_1-a_2} d_2^{a_2-a_3} \cdots d_{k-1}^{a_{k-1}-a_k} a_k^{a_k} ,$$

where

$$d_t = \det \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1t} \\ p_{211} & p_{22} & \cdots & p_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ p_{t1} & p_{t2} & \cdots & p_{tt} \end{pmatrix} ,$$

$a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are integers. The corresponding irreducible representation of $Gl_k \times Gl_k$ is isomorphic to $\rho_k^D \otimes (\rho_k^D)^*$ where $D = (a_1, a_2, \ldots, a_k)$.

Thus in this case, our main result can be rephrased as

Theorem 1(B). *For any integer $n \geq 0$, the polynomial function*

$$(p_{11} + p_{22} + \cdots + p_{kk})^n$$

has a nonzero projection to all $\rho_k^D \otimes (\rho_k^D)^*$ with $|D| = n$. Moreover, these projections span linearly the space of $Gl_k(\mathbb{C})$-invariants in $P^n(M_k)$.

(C) $G = Gl_{2k}(\mathbb{C})$, $V = \Lambda_k$. It is well known ([H1, Sh], etc.) that

$$P(\Lambda_k) \cong \sum_{D \text{ has even columns}} \rho_{2k}^D ;$$

here we say D has even columns if D is of the form $D = (a_1, a_1, a_2, a_2, \dots, a_k, a_k)$.

In fact, one has explicit descriptions of all the highest weight vectors, they are

$$Pf_1^{a_1-a_2} Pf_2^{a_2-a_3} \cdots Pf_{k-1}^{a_{k-1}-a_k} Pf_k^{a_k},$$

where $Pf_i(p)$ is the ith principal Pfaffian of $p \in \Lambda_k$, and it satisfies $Pf_i^2 = d_{2i}$, the $2i$th principal minor [W]. The corresponding irreducible representation of Gl_{2k} has the highest weight $(a_1, a_1, a_2, a_2, \dots, a_k, a_k)$.

Thus in this case, Theorem 1 can be rephrased as

Theorem 1(C). *For any integer $n \geq 0$, the polynomial function*

$$(p_{1,2} + p_{3,4} + \cdots + p_{2k-1,2k})^n$$

has a nonzero projection to all ρ_{2k}^D such that D has even columns and $|D| = 2n$. Moreover, these projections span linearly the space of $Sp_{2k}(\mathbb{C})$-invariants in $P^n(\Lambda_k)$.

V. Final remarks

Theorems 1(A), 1(B), and 1(C) are related to the theory of dual pairs and invariant distributions of classical groups. See the author's Yale thesis [Zhu]. In fact, with appropriate choices of λ, $\lambda \neq 0$, $L(\lambda) = e^{\lambda\phi}$ can be interpreted as some kind of Lebesgue measure and Dirac distribution. In that context, the refined Theorem 1′ implies the existence of certain invariant distributions. The author is quite surprised to see that the indicated existence result is purely algebraic as demonstrated in this note.

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