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CESÀRO AND GENERAL EULER-BOREL SUMMABILITY

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ABSTRACT. The general Euler-Borel summability method is a method that includes the Euler, discrete Borel, Meyer-König, Taylor, and Karamata methods as special cases. We prove that under a certain condition the Cesàro summability of a sequence implies its summability by a general Euler-Borel method.

1. INTRODUCTION

We will give a condition under which the Cesàro summability of a sequence implies its summability by a general Euler-Borel summability method, which is also called a Sonnenschein method. The definitions of these methods are as follows.

A sequence (s_k) is summable to S by the Cesàro method C_r $(r > -1)$ if

$$\frac{r!}{n^r} s_n^r \to S \quad \text{as } n \to \infty,$$

where

$$s_n^r = \sum_{k=0}^{n} \binom{n-k+r-1}{n-k} s_k .$$

(s_k) is summable to S by the general Euler-Borel method (E, f), where f is a function analytic at the origin, if $(a_{n,k})$ satisfies

$$(f(z))^n = \sum_{k=0}^{\infty} a_{n,k} z^k, \qquad n = 0, 1, \ldots$$

and

$$\sum_{k=0}^{\infty} a_{n,k} s_k \to S \quad \text{as } n \to \infty .$$

$|z|$ is of course assumed to be small enough in the equation defining $a_{n,k}$.

The Cesàro method is well known. The general Euler-Borel method has been studied in [1, 3, 6], among others. Examples of the method include the Euler method E_q, $q > 0$, the discrete Borel method, the Meyer-König method S_r,

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$0 < r < 1$, the Taylor method T_r, $0 < r < 1$, and the Karamata method. The functions defining these methods are, respectively,

(1) $f(z) = (z+q)/(1+q)$.
(2) $f(z) = \exp(z-1)$.
(3) $f(z) = (1-r)/(1-rz)$.
(4) $f(z) = (1-r)z/(1-rz)$.
(5) $f(z) = (\alpha+(1-\alpha-\beta)z)/(1-\beta z)$, where $\alpha < 1$, $\beta < 1$, and $\alpha+\beta > 0$.

A summability method is called regular if it sums each convergence sequence to its limit. The following theorem is due to B. Bajšanski [1].

Theorem A. *Suppose that*

(1) f *is analytic for* $|z| < R$, $R > 1$;
(2) $|f(z)| < 1$ *for* $|z| \le 1$, $z \neq 1$;
(3) $f(1) = 1$; *and*
(4) *the number* A *defined by*

$$f(z) - z^\alpha = Ai^p(z-1)^p + o(1)(z-1)^p, \qquad \alpha = f'(1),\ A \neq 0,\ \text{as } z \to 1$$

satisfies $\Re A \neq 0$.

Then the method (E, f) *is regular.*

Condition 4 may seem to be complicated, but J. Clunie and P. Vermes have proved in [3] that if conditions 1–3 hold, then condition 4 is also necessary for the regularity of the method. The above results of Bajšanski and Clunie and Vermes were rediscovered by D. Newman [5], with a considerably shorter proof.

If (E, f) satisfies the conditions of Theorem A then we will denote the number p in condition 4 by $p(f)$.

It is proved in [1] that $p(f)$ is necessarily an even integer, $\Re A < 0$, and $\alpha > 0$.

All the above examples satisfy the conditions of Theorem A, with $p(f) = 2$. Moreover, in [6] it is shown that for each integer p there is a method (E, f) satisfying the conditions of Theorem A with $p(f) = p$.

The following example shows that in general the C_r summability of a sequence does not imply its (E, f) summability:

$$\sum_{k\ge 1} k^{-1/p} \exp(\beta i k^{1-1/p}), \qquad \text{where } \beta > 0 \text{ and } p > 1.$$

This series is summable C_r for every $r > 0$ but is divergent. (See [4, p. 213].) Since β is real,

$$|k^{-1/p} \exp(\beta i k^{1-1/p})| = k^{-1/p}.$$

Since the series is divergent, the following Tauberian theorem shows that it is not (E, f) summable if $p(f) = p$.

Theorem B [6, Corollary to Theorem 3]. *If a sequence* (s_k) *is summable by a method* (E, f) *satisfying the conditions of Theorem* A *and*

$$s_k - s_{k-1} = O(k^{-1/p(f)}),$$

then (s_k) *is convergent.*

However, we have the following

Theorem. *Suppose that r is a positive integer. Suppose that a sequence (s_k) satisfies*

$$\frac{r!}{n^r} s_n^r = S + o(n^{r(-1+1/p)}) \quad \text{as } n \to \infty,$$

where p is an even integer. Suppose that (E, f) is a method satisfying the conditions of Theorem A with $p(f) = p$. Then (s_k) is summable by (E, f) to S.

If $r = 1$, $p = 2$, and (E, f) is an Euler method, then this theorem reduces to a result of K. Knopp. See [4, Theorem 149]. The special case of the theorem with $p = 2$ and (E, f) equal to an Euler method, a Meyer-König method, or a Taylor method is due to D. Borwein and T. Markovich [2, Theorem 3]. They also proved that if $p = 2$ then we may replace the method (E, f) by a Borel-type method or a Valiron method in the theorem. See [2, Theorems 1, 2].

2. Proof of the theorem

We will denote constants by K, not necessarily the same at each occurrence.
Without loss of generality we may assume that $S = 0$, so that

$$(1) \qquad s_n^r = o(n^{r/p}) \quad \text{as } n \to \infty.$$

Hence $s_k^r = \varepsilon_k k^{r/p}$, where $\varepsilon_k = o(1)$ as $k \to \infty$.
Let $\delta_k = \sup_{n \geq k} |\varepsilon_n|$. Then $\delta_k = o(1)$ and is decreasing. Also, we have

$$(2) \qquad |s_k^r| \leq \delta_k k^{r/p}.$$

Let (E, f) be a method satisfying the conditions of Theorem A with matrix $(a_{n,k})$ and $p(f) = p$. We will divide the proof of

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} s_k = 0$$

into nine steps. The first step is similar to that of the proof of Theorem 2 in [2].

I. If $n \geq 1$ then $\sum_{k=0}^{\infty} a_{n,k} s_k = \sum_{k=0}^{\infty} (\Delta^r a_{n,k}) s_k^r$, where

$$\Delta^r a_{n,k} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j a_{n,k+j}.$$

Proof. Let $m > r$. Applying Abel's partial summation formula r times we have

$$\sum_{k=0}^{m} a_{n,k} s_k = \sum_{k=0}^{m-r} (\Delta^r a_{n,k}) s_k^r + \sum_{k=0}^{r-1} (\Delta^k a_{n,m-k}) s_{m-k}^{k+1}.$$

We will show that each term in the second sum tends to 0 as $m \to \infty$. Then the equality to be proved will follow.

Since (s_k) is C_r summable, by [4, Theorem 46], $s_{m-k}^{k+1} = o(m^r)$ for $0 \leq k < r-1$. This is also true for $k = r-1$, by (1) and the fact that $p > 1$. Since $n \geq 1$ and $\sum_{k=0}^{\infty} a_{n,k} z^k$ converges for some z with $|z| > 1$, $a_{n,k} z^k \to 0$ as $k \to \infty$, with $|z| > 1$. Hence $(\Delta^k a_{n,m-k}) s_{m-k}^{k+1} = o(1)$ as $m \to \infty$, for $0 \leq k \leq r-1$.

This completes the proof.

Thus to prove the theorem if suffices to prove that

$$\sum_{k=0}^{\infty}(\Delta^r a_{n,k})s_k^r \to 0 \quad \text{as } n \to \infty.$$

II. $\Delta^r a_{n,k} = (1/2\pi i)\int_{|z|=a}(f(z))^n z^{-k-1}z^{-r}(z-1)^r\,dz$, where a is an number in $(0, R)$, where in turn R is the number in condition 1 of Theorem A.

Proof. We have

$$\begin{aligned}\Delta^r a_{n,k} &= \sum_{j=0}^{r}\binom{r}{j}(-1)^j a_{n,k+j}\\ &= \sum_{j=0}^{r}\binom{r}{j}(-1)^j \frac{1}{2\pi i}\int_{|z|=a}(f(z))^n z^{-k-j-1}\,dz,\end{aligned}$$

by Cauchy's integral formula,

$$\begin{aligned}&= \frac{1}{2\pi i}\int_{|z|=a}(f(z))^n z^{-k-1}z^{-r}\sum_{j=0}^{r}\binom{r}{j}(-1)^j z^{r-j}\,dz\\ &= \frac{1}{2\pi i}\int_{|z|=a}(f(z))^n z^{-k-1}z^{-r}(z-1)^r\,dz.\end{aligned}$$

This proves II.

III. $|\sum_{k=0}^{\infty}(\Delta^r a_{n,k})s_k^r| \le K(T_n + U_n)$, where

$$T_n = \sum_{k\le\alpha n}\delta_k k^{r/p}\int_{|z|=R_1}|f(z)|^n|z|^{-k-1}|z-1|^r|dz|,$$

$$U_n = \sum_{k>\alpha n}\delta_k k^{r/p}\int_{|z|=R_2}|f(z)|^n|z|^{-k-1}|z-1|^r|dz|,$$

where $\alpha = f'(1)$, $R_1 = 1 - n^{-1/p}$, and $R_2 = 1 + n^{-1/p}$. We assume that n is so large that $R_2 < R$, the number in condition 1 of Theorem A.

Proof. In the integral in II let

$$a = \begin{cases} R_1 = 1 - n^{-1/p} & \text{if } k \le \alpha n,\\ R_2 = 1 + n^{-1/p} & \text{if } k > \alpha n.\end{cases}$$

Since $|z|^{-r}$ is bounded on $|z| = R_j$, $j = 1, 2$, we have for $k = 0, 1, \ldots$

$$|\Delta^r a_{n,k}| \le K\int_{|z|=R_j}|f(z)|^n|z|^{-k-1}|z-1|^r|dz|,$$

where $j = 1$ if $k \le \alpha n$ and $j = 2$ if $k > \alpha n$.

III follows from these inequalities and (2).

We note that

$$T_n = \sum_{k\le\alpha n}\delta_k k^{r/p}\int_0^{2\pi}|f(R_1e^{it})|^n R_1^{-k}|R_1e^{it} - 1|^r\,dt,$$

$$U_n = \sum_{k>\alpha n}\delta_k k^{r/p}\int_0^{2\pi}|f(R_2e^{it})|^n R_2^{-k}|R_2e^{it} - 1|^r\,dt.$$

IV. By condition 2 of Theorem A and given $0 < \delta < 1$, there exists $\varepsilon > 0$ and $N_\delta > 0$ such that for $n > N_\delta$ and $t \in [\varepsilon, 2\pi - \varepsilon]$, we have

$$|f(R_j e^{it})| < 1 - \delta < 1, \qquad j = 1, 2.$$

We fix the numbers δ, ε, and N_δ.

V. $T_n = V_n + o(1)$ as $n \to \infty$, where

$$V_n = \sum_{k \le \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})|^n R_1^{-k} |R_1 e^{it} - 1|^r \, dt.$$

Proof. We have to prove that

$$\sum_{k \le \alpha n} \delta_k k^{r/p} \int_{\varepsilon}^{2\pi - \varepsilon} |f(R_1 e^{it})|^n R_1^{-k} |R_1 e^{it} - 1|^r \, dt = o(1) \quad \text{as } n \to \infty.$$

Since δ_k and $|R_1 e^{it} - 1|^r$ are bounded, if $n > N_\delta$ then by IV, the quantity on the left

$$\begin{aligned}
&\le K \sum_{k \le \alpha n} k^{r/p} \int_{\varepsilon}^{2\pi - \varepsilon} (1 - \delta)^n R_1^{-k} \, dt \\
&\le K (1 - \delta)^n \sum_{k \le \alpha n} k^{r/p} (1 - n^{-1/p})^{-k} \\
&\le K (1 - \delta)^n n^{r/p} \sum_{k \le \alpha n} (1 - n^{-1/p})^{-k} \\
&\le K (1 - \delta)^n n^{r/p} \alpha n (1 - n^{-1/p})^{-\alpha n} \\
&= o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Using IV again, we can prove

VI. $U_n = W_n + o(1)$ as $n \to \infty$, where

$$W_n = \sum_{k > \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_2 e^{it})|^n R_2^{-k} |R_2 e^{it} - 1|^r \, dt.$$

We will omit the details.

It remains to show that $V_n = o(1)$ and $W_n = o(1)$ as $n \to \infty$. To do so we need the following generalization of Lemma 1 in [1], where $r = 0$.

VII. $\int_{-\varepsilon}^{\varepsilon} |f(R_j e^{it}) R_j^{-\alpha}|^n |R_j e^{it} - 1|^r \, dt = O(n^{-(r+1)/p})$ as $n \to \infty$, $j = 1$ or 2.

Proof. We will only prove that

$$\int_0^{\varepsilon} |f(R_1 e^{it}) R_1^{-\alpha}|^n |R_1 e^{it} - 1|^r \, dt = O(n^{-(r+1)/p}) \quad \text{as } n \to \infty,$$

the rest of the proof being similar.

It follows from condition 4 of Theorem A that

$$|f(re^{it}) r^{-\alpha}| = 1 + O(1)(re^{it} - 1)^p \quad \text{as } t \to 0, r \to 1.$$

Let $\psi(r, t) = \log |f(re^{it}) r^{-\alpha}|$. Then we have

$$\psi(r, t) = \log(1 + O(1)(re^{it} - 1)^p) = O(1)(re^{it} - 1)^p$$

as $t \to 0$, $r \to 1$. Therefore all partial derivatives of ψ of order $< p$ are equal to 0 at $r = 1$, $t = 0$.

Next we estimate $(\partial^p/\partial t^p)\psi(r, t)$ in a neighborhood of $r = 1$, $t = 0$.

By condition 4 of Theorem A again, we have

$$f(e^{it})e^{-i\alpha t} = 1 + Ai^p(e^{it} - 1)^p + o(1)(e^{it} - 1)^p \quad \text{as } t \to 0.$$

Hence

$$|f(e^{it})| = 1 + i^{2p}\Re A t^p + o(t^p) = 1 + \Re A t^p + o(t^p) \quad \text{as } t \to 0,$$

since p is even.

Thus

$$\psi(1, t) = \log|f(e^{it})| = \log(1 + \Re A t^p + o(t^p)) = \Re A t^p + o(t^p) \quad \text{as } t \to 0.$$

This implies that $(\partial^p/\partial t^p)\psi(1, 0) = (p!)\Re A$. Since $\Re A < 0$, if r is close to 1 and $|t| < \varepsilon$ then we have

$$\frac{1}{p!}\frac{\partial^p}{\partial t^p}\psi(r, t) < -M$$

for some positive constant M. (We may need to decrease ε, which we have fixed in step IV, but this does no harm.)

By Taylor's formula,

$$\psi(r, t) = \sum_{m=0}^{p-1} C_m(r)t^m + C_p(r, t)t^p, \tag{3}$$

where

$$C_m(r) = \frac{1}{m!}\left.\frac{\partial^m \psi(r, t)}{\partial t^m}\right|_{t=0} \quad \text{for } m = 0, 1, \dots, p-1,$$

$$C_p(r, t) = \frac{1}{p!}\left.\frac{\partial^p \psi(r, t)}{\partial t^p}\right|_{t=\tau}, \qquad |\tau| < |t| < \varepsilon.$$

Thus $C_p(r, t) < -M$ if r is close to 1 and $|t| < \varepsilon$.

Since all partial derivatives of ψ of order $< p$ vanish at $r = 1$, $t = 0$, and since

$$\frac{d^n C_m(r)}{dr^n} = \frac{1}{m!}\left.\frac{\partial^{n+m}\psi(r, t)}{\partial r^n \partial t^m}\right|_{t=0},$$

all derivatives of $C_m(r)$ of order $< p - m$ vanish at $r = 1$. Hence $r = 1$ is a zero of order at least $p - m$ of $C_m(r)$ and there exist constants $K_m > 0$ such that

$$C_m(r) \le K_m|r - 1|^{p-m}$$

for $m = 0, 1, \dots, p-1$, if r is close to 1. It now follows from (3) and the estimates of $C_p(r, t)$ and $C_m(r)$ that if $|t| < \varepsilon$ and n is large enough, so that $R_1 = 1 + n^{-1/p}$ is close enough to 1, then

$$\begin{aligned}\psi(R_1, t) &\le \sum_{m=0}^{p-1} K_m|R_1 - 1|^{p-m}t^m - Mt^p \\ &\le \sum_{m=0}^{p-1} K_m n^{-1+m/p}t^m - Mt^p.\end{aligned}$$

On the other hand, an easy computation shows that

$$|R_1 e^{it} - 1|^r \le n^{-r/p}(1 + 2n^{2/p}(1 - \cos t))^{r/2}.$$

Finally, we have

$$\begin{aligned}
&\int_0^\varepsilon |f(R_1 e^{it}) R_1^{-\alpha}|^n |R_1 e^{it} - 1|^r \, dt \\
&= \int_0^\varepsilon \{\exp(n\psi(R_1, t)\} |R_1 e^{it} - 1|^r \, dt \\
&\le n^{-r/p} \int_0^\varepsilon \left\{ \exp\left(n \left(\sum_{m=0}^{p-1} K_m n^{-1+m/p} t^m - M t^p \right) \right) \right\} (1 + 2n^{2/p}(1 - \cos t))^{r/2} \, dt \\
&\le n^{-(r+1)/p} \int_0^{\varepsilon n^{1/p}} \left\{ \exp\left(\sum_{m=0}^{p-1} K_m v^m - M v^p \right) \right\} (1 + 2n^{2/p}(1 - \cos n^{-1/p} v))^{r/2} \, dv
\end{aligned}$$

(We have made the substitution $v = n^{1/p} t$.)

$$= O(n^{-(r+1)/p}) \quad \text{as } n \to \infty.$$

This completes the proof of VII.

VIII. $V_n = o(1)$ as $n \to \infty$.

Proof. We have

$$\begin{aligned}
V_n &= \sum_{k \le \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})|^n R_1^{-k} |R_1 e^{it} - 1|^r \, dt \\
&= \sum_{k \le \alpha n} \delta_k k^{r/p} R_1^{\alpha n - k} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it}) R_1^{-\alpha}|^n |R_1 e^{it} - 1|^r \, dt \\
&\le K \sum_{k \le \alpha n} \delta_k k^{r/p} R_1^{\alpha n - k} n^{-(r+1)/p}, \qquad \text{by VII}, \\
&\le K n^{-(r+1)/p} \sum_{k \le \alpha n} \delta_k k^{r/p} (1 - n^{-1/p})^{\alpha n - k},
\end{aligned}$$

since $R_1 = 1 - n^{-1/p}$.

Let $c > 0$. Since δ_k tends to 0, there exists $N_c > 0$ such that if $k > N_c$, then $\delta_k < c$. Hence if $\alpha n > N_c$ then

$$\begin{aligned}
V_n \le K n^{-(r+1)/p} (1 - n^{-1/p})^{\alpha n} &\sum_{k \le N_c} \delta_k k^{r/p} (1 - n^{-1/p})^{-k} \\
+ K n^{-(r+1)/p} &\sum_{N_c < k \le \alpha n} c k^{r/p} (1 - n^{-1/p})^{\alpha n - k}.
\end{aligned}$$

For each fixed $c > 0$, the first term on the right

$$= K n^{-(r+1)/p} (1 - n^{-1/p})^{\alpha n} O(1) = o(1) \quad \text{as } n \to \infty.$$

The second term

$$\le Kn^{-(r+1)/p} cn^{r/p} \sum_{N_c<k\le \alpha n} (1-n^{-1/p})^{\alpha n-k}$$

$$\le Kcn^{-1/p} \sum_{m=0}^{\infty} (1-n^{-1/p})^m$$

$$\le Kcn^{-1/p} \frac{1}{1-(1-n^{-1/p})} = Kc.$$

Since c is an arbitrary positive number, $V_n = o(1)$ as $n \to \infty$.

IX. $W_n = o(1)$ as $n \to \infty$.

Proof. Since δ_k is decreasing,

$$W_n \le \delta_{[\alpha n]} \sum_{k>\alpha n} k^{r/p} R_2^{\alpha n-k} \int_{-\varepsilon}^{\varepsilon} |f(R_2 e^{it}) R_2^{-\alpha}|^n |R_2 e^{it} - 1|^r \, dt$$

$$\le K\delta_{[\alpha n]} n^{-(r+1)/p} \sum_{k>\alpha n} k^{r/p} R_2^{\alpha n-k}, \qquad \text{by VII}.$$

Let $m = [\alpha n]$ and $R = R_2^{-1}$. Then we have

$$W_n \le K\delta_{[\alpha n]} n^{-(r+1)/p} R_2^{\alpha n} \sum_{k\ge m} k^{r/p} R^k.$$

We will prove that

$$n^{-(r+1)/p} R_2^{\alpha n} \sum_{k\ge m} k^{r/p} R^k = O(1) \tag{4}$$

and $n \to \infty$. Then since $\delta_n = o(1)$, IX follows.

Let q be the smallest integer satisfying $q \ge r/p$. Then the sum in (4)

$$\le Kn^{r/p-q} \sum_{k\ge m} k^q R^k$$

$$\le Kn^{r/p-q} \sum_{k\ge m} (k+1)(k+2)\cdots(k+q) R^k$$

$$= Kn^{r/p-q} \frac{d^{(q)}}{dR^{(q)}} \sum_{k\ge m} R^{k+q}$$

$$= Kn^{r/p-q} \frac{d^{(q)}}{dR^{(q)}} \frac{R^{m+q}}{1-R}$$

$$= Kn^{r/p-q} \sum_{j=0}^{q} \binom{q}{j} \left(\frac{d^{(q-j)}}{dR^{(q-j)}} R^{m+q} \right) \left(\frac{d^{(j)}}{dR^{(j)}} \frac{1}{1-R} \right)$$

$$= Kn^{r/p-q} \sum_{j=0}^{q} \binom{q}{j} (m+q)(m+q-1)\cdots(m+j+1) R^{m+j} \frac{j!}{(1-R)^{j+1}}.$$

Since

$$(m+q)(m+q-1)\cdots(m+j+1) \le Kn^{q-j},$$

and

$$\frac{1}{(1-R)^{j+1}} \leq K n^{(j+1)/p},$$

the sum in (4)

$$\leq K n^{r/p-q} \sum_{j=0}^{q} \binom{q}{j} n^{q-j} R^{m+j} (j!) n^{(j+1)/p} \leq K R^m n^{(r+1)/p}.$$

Since $R = R_2^{-1}$, this implies that the quantity on the left of (4) is bounded. This proves IX.

The proof of the theorem is complete.

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