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# PROJECTIONS $P$ ON $C = C[-1, 1]$ WHICH INTERPOLATE AT $\dim(P(C))$ OR MORE POINTS

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ABSTRACT. Let $V$ be an $n$ dimensional subspace of $C[-1, 1]$. This paper gives a necessary and sufficient condition for a bounded linear projection $P$ from $C[-1, 1]$ onto $V$ to have the property that $Pf$ interpolates $f$ at $n$ or more points for any $f \in C[-1, 1]$.

## 1. INTRODUCTION

Let $C[-1, 1]$ denote the Banach space of all continuous functions defined on the real interval $[-1, 1]$ and $V = \text{span}\{v_1, v_2, \dots, v_n\}$ be an $n$ dimensional subspace of $C[-1, 1]$. Let $P$ be a bounded linear projection from $C[-1, 1]$ onto $V$. For $f \in C[-1, 1]$, $Pf$ can be represented uniquely by

$$Pf = \sum_{i=1}^{n} a_i(f) v_i, \tag{1}$$

where $a_i(f)$ is a bounded linear functional on $C[-1, 1]$. By the Riesz representation theorem there exists a Borel signed meausre $\mu_i$ on $[-1, 1]$ such that

$$a_i(f) = \int_{-1}^{1} f\, d\mu_i$$

for each $i$ and hence

$$Pf = \sum_{i=1}^{n} \left( \int_{-1}^{1} f\, d\mu_i \right) v_i. \tag{2}$$

By the uniqueness of (1) we obtain

$$\int_{-1}^{1} v_i\, d\mu_j = \delta_{ij}. \tag{3}$$

It is easy to see that if we consider another basis for $V$, then another set of $n$ measures, each of which is a linear combination of $\{\mu_1, \dots, \mu_n\}$, will be so generated.

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When $\{\mu_i\}_{i=1,n}$ are point evaluation measures, $P$ is an interpolation projection. In this case $Pf$ interpolates any $f \in C[-1, 1]$ at $n$ fixed points. In this paper we shall study the question of characterizing those projections that interpolate each $f$ at $n$ or more points. Throughout this paper $P$ will always denote a bounded linear projection from $C[-1, 1]$ onto $V$ and we shall omit $-1$ and 1, the lower and upper limits of the integral, for the sake of simplicity. Indeed, any results in this paper can be easily extended to any interval.

## 2. A KNOWN SUFFICIENT CONDITION

In [1] a sufficient condition for a projection $P$ with $Pf$ interpolating $f$ at $n$ or more points is given. We start by describing and discussing this result.

**Definition 2.1.** A set of $n$ Borel measures $\{\mu_1, \dots, \mu_n\}$ is said to be positive-separated if each $\mu_i$ is nonnegative and $\text{supp}(\mu_i) \leq \text{supp}(\mu_j)$ or $\text{supp}(\mu_j) \leq \text{supp}(\mu_i)$ for $i \neq j$, where $\text{supp}(\mu)$ denotes the support of $\mu$ and $X \leq Y$ means $x \leq y$ for any $x \in X$ and $y \in Y$. A projection $P$ is said to be positive-separated if there exist $n$ measures $\{\mu_1, \dots, \mu_n\}$ such that they are positive-separated and (2) holds.

**Theorem 2.2** [1]. *If $P$ is positive-separated, then $Pf$ interpolates $f$ at $n$ or more points (counting multiplicity) for any $f \in C[-1, 1]$.*

Here the multiplicity means that we count two interpolating points at $x \in (-1, 1)$ if $(Pf - f)(x) = 0$ and $Pf - f$ does not change the sign at $x$. The proof of Theorem 2.2 in [1] had a minor error. Thus we give a proof here and we actually prove a slightly stronger result as stated below.

**Theorem 2.3.** *If $P$ is positive-separated, then $Pf$ interpolates $f$ at $n$ or more distinct points for any $f \in C[-1, 1]$.*

*Proof.* From (2) and (3) we have

$$\int Pf\,d\mu_j = \sum_{i=1}^{n} \int f\,d\mu_i \int v_i\,d\mu_j = \int f\,d\mu_j. \tag{4}$$

This implies

$$\int (pf - f)\,d\mu_j = 0, \qquad j = 1, \dots, n.$$

Since $\mu_j$ is nonnegative, there exists $x_j \in \text{supp}(\mu_j)$ such that

$$(pf - f)(x_j) = 0, \qquad j = 1, \dots, n.$$

Next we show that we can pick these points to be distinct. Let $[a_j, b_j]$ be the smallest closed interval containing $\text{supp}(\mu_j)$. If $\mu_j(a_j, b_j) > 0$, then we can pick $x_j \in (a_j, b_j)$. Excluding these $[a_j, b_j]$, there may remain several groups of $[a_j, b_j]$ each of which satisfies

$$a_i \leq b_i = a_{i+1} \leq \cdots \leq b_{i+k-1} = a_{i+k} \leq b_{i+k}$$

and

$$(Pf - f)(x) \neq 0, \qquad \text{for } x \in (a_j, b_j),\ i \leq j \leq i + k.$$

Now we claim that we can pick $k+1$ distinct interpolating points in $[a_i, b_{i+k}]$. Indeed, if $k = 0$, nothing needs to be proven. For $k \geq 1$, if $\mu_i(\{a_i\}) > 0$ or

$\mu_{i+k}(\{b_{i+k}\}) > 0$, then either $(Pf - f)(a_i) = 0$ or $(Pf - f)(b_{i+k}) = 0$ and we can take off one interval and reduce to the $k-1$ case. Otherwise,

$$\bigcup_{j=i}^{i+k} \operatorname{supp}(\mu_j) \subset \{a_{i+1}, \dots, a_{i+k}\}$$

and the $k+1$ measures, $\{\mu_i \dots, \mu_{i+k}\}$, must be linearly dependent which contradicts (3).

This theorem includes the point evaluation projection case (interpolating polynomial case), but, unfortunately, neither the nonnegativeness nor separated assumption is necessary for $n \geq 2$. The following example shows this.

**Example 2.4.** Let $n = 2$, $V = \operatorname{span}\{1, x\}$, $v_1 = (1/2) - x$ and $v_2 = (2/\pi)x$. Let $\mu_1$ and $\mu_2$ be defined by

$$\int f \, d\mu_1 = \int f(x) \, dx$$

and

$$\int f \, d\mu_2 = \int \left(x/\sqrt{1-x^2}\right) f(x) \, dx.$$

It is easy to verify that $P$ defined by

$$Pf = \left(\int f \, d\mu_1\right) v_1 + \left(\int f \, d\mu_2\right) v_2$$

is a linear projection on $V$ and $\int v_i \, d\mu_j = \delta_{ij}$. For any linear combination $\mu = \alpha\mu_1 + \beta\mu_2$, $\operatorname{supp}(\mu) = [-1, 1]$ and $\mu$ is not nonnegative unless $\beta = 0$, but $Pf$ interpolates $f$ at two or more points (this follows from our main result Theorem 3.3 below).

## 3. Necessary and sufficient conditions

First, we give an easily obtained but useful, necessary, and sufficient condition.

**Theorem 3.1.** *A linear projection $P$ will have $Pf$ interpolating $f$ at $n$ or more points for any $f \in C[-1, 1]$, if and only if for any $f \in \{f \in C[-1, 1] | Pf = 0\}$, the null space of $P$, $f$ must have at least $n$ zeros.*

*Proof.* ($\Rightarrow$) Evidently.

($\Leftarrow$) Suppose there exists $f \in C[-1, 1]$ such that $Pf$ interpolates $f$ at less than $n$ points. Then $Pf - f \in \{f \in C[-1, 1] | Pf = 0\}$ and $Pf - f$ does not have $n$ zeros.

Theorem 3.1 is true whether or not the multiplicity is considered. To give a more significant necessary and sufficient condition we need the definition of $WT$ measure vector spaces which has been introduced and studied in [4].

**Definition 3.2.** Let $\mu_i$, $i = 1, \dots, n$, be $n$ finite Borel signed measures on $D = [-1, 1]$. $M = \operatorname{span}\{\mu_1, \dots, \mu_n\}$ is a $WT$ measure vector space means for any $x_1 < x_2 < \dots < x_{n-1}$ in $D$ there exists a nontrivial $\mu \in M$ such that

(5) $\quad (-1)^i \mu|_{[x_{i-1}, x_i]} \geq 0, \; i = 1, \dots, n,$ and $\mu(\{x_i\}) = 0, \; i = 1, \dots, n-1$

where $x_0 = -1$ and $x_n = 1$.

*Remark*. The definition of $WT$ measure vector spaces given in [4] is different from the one above, namely, $M$ is a $WT$ measure vector space if any $\mu \in M$ has at most $n-1$ sign changes. However, a theorem in [4] shows they are equivalent. Here we use this definition simply to avoid referring to a preprint theorem.

Now we state our main result.

**Theorem 3.3.** *Let $P$ be a linear projection from $C[-1,1]$ onto an $n$ dimensional subspace $V = \text{span}\{v_1, \dots, v_n\}$ and $\mu_i$, $i = 1, \dots, n$, be defined as in* (2). *Then $Pf$ interpolates $f$ at $n$ or more points* (*including multiplicity*) *for any $f \in C[-1,1]$, if and only if $M = \text{span}\{\mu_1, \dots, \mu_n\}$ is a $WT$ measure vector space.*

The proof will be given in the next section. Here we point out that this necessary and sufficient condiition depends only on $\mu_i$, $i = 1, \dots, n$. In other words, any projection $P$ onto $H = \text{span}\{h_1, \dots, h_n\}$ defined by

$$Pf = \sum_{i=1}^{n} \left( \int f\, d\mu_i \right) h_i$$

has this property if $M$ is a $WT$ measure vector space.

From the definition, it is easy to see that if $\text{span}\{g_1, \dots, g_n\}$ is a weak Chebyshev space and $\mu_1, \dots, \mu_n$ are defined by

$$\mu_i(A) = \int_A g_i\, dx, \qquad i = 1, \dots, n,$$

then $M = \text{span}\{\mu_1, \dots, \mu_n\}$ is a $WT$ measure vector space. Hence the space spanned by $\mu_1$ and $\mu_2$ defined in Example 2.4 is a $WT$ measure vector space. Therefore by Theorem 3.3 the projection $P$ defined in the example has the property that $Pf$ interpolates $f$ at two or more points for any $f \in C[-1,1]$.

## 4. The proof of Theorem 3.3

We need three lemmas to begin.

**Lemma 4.1.** *Let $D = [-1,1]$ and $M = \text{span}\{\mu_1, \dots, \mu_n\}$ be a $WT$ measure vector space. Then, for any $x_1, \dots, x_k, x_{k+1}, \dots, x_m$, in $D$ with $k + 2(m-k) \le n-1$ and being reindexed as $y_1 < y_2 < \cdots < y_m$, there exists a nontrivial measure $\mu \in M$ such that*

(6)
$$\sigma_i \mu|_{(y_i, y_{i+1})} \ge 0, \qquad i = 0, 1, \dots, m, \quad \text{and} \quad \mu(\{y_i\}) = 0, \qquad i = 1, \dots, m,$$

*where $y_0 = -\infty$, $y_{m+1} = +\infty$, $\sigma_0 = 1$, $\sigma_{i+1} = \lambda_i \sigma_i$, and $\lambda_i = -1$ if $y_{i+1} \in \{x_1, \dots, x_k\}$ and $\lambda_i = 1$ if $y_{i+1} \in \{x_{k+1}, \dots, x_m\}$.*

*Proof*. We first assume $-1 < y_1$ and $y_m < 1$. Choose $m-k$ points $x_j + \varepsilon$, $j = k+1, \dots, m$, such that $(x_j, x_j + \varepsilon) \cap \{x_1, \dots, x_m\} = \varnothing$, and another $n-k-1-2(m-k)$ points $1 - i\varepsilon$, $i = 1, \dots, n-k-1-2(m-k)$, such that $y_m + \varepsilon < 1 - \varepsilon(n-k-1-2(m-k))$. Rewrite them and $y_1, \dots, y_m$ as $-1 < z_1 < z_2 < \cdots < z_{n-1} < 1$. Then by Definition 3.2 there exists a nontrivial $\mu_\varepsilon \in M$ satisfying (5) with $x_i$ replaced by $z_i$. We can also require $\|\mu_i\| = 1$. By the compactness of a bounded set in a finite dimensional space there is a sequence $\{\mu_{\varepsilon_i}\}$ that converges to a $\mu \in M$ as $\varepsilon_i \to 0$. It is easy to see this $\mu$

satisfies (6) and $\|\mu\| = 1$. If $y_1 = -1$ and (or) $y_m = 1$, we can extend the underlying set $[-1, 1]$ to $[-2, 2]$ with $\mu(A) = 0$ for any $\mu \in M$ and any $A \subset [-2, -1) \cup (1, 2]$.

We denote the point evaluation measure at $x$ as $\mu_x$, i.e.,

$$\mu_x(E) = \begin{cases} 1 & x \in E, \\ 0 & x \notin E. \end{cases}$$

**Lemma 4.2.** *Let $D = [-1, 1]$ with the standard topology and let $\mu_1, \dots, \mu_n$ be $n$ finite Borel signed measures on $D$. Then the following two statements are equivalent:*

(1) *$M = \text{span}\{\mu_1, \dots, \mu_n\}$ is a $WT$ measure vector space.*

(2) *For every $x_1 < \cdots < x_{n-1}$ in $D$, there exists a nontrivial $\mu \in M$ such that*

$$(7) \qquad (-1)^i \mu|_{(x_{i-1}, x_i)} \geq 0, \qquad i = 1, \dots, n$$

*where $x_0 = -\infty$ and $x_n = \infty$, and*

$$(8) \qquad \begin{aligned} \mu(\{x_i\}) &= 0 \quad \text{if } \mu_{x_i} \in M, \\ \mu(\{x_1\}) &\geq 0 \quad \text{if } x_1 = -1, \\ (-1)^n \mu(\{x_{n-1}\}) &\leq 0 \quad \text{if } x_{n-1} = 1. \end{aligned}$$

*Proof.* $(1 \Rightarrow 2)$ It is obvious.

$(2 \Rightarrow 1)$ We prove this direction by induction. Suppose the following statement is true for $m \geq k$.

$M$ is a $WT$ measure vector space, if and only if (2) holds and, in addition, $\mu(\{x_i\}) = 0$ for $i = 1, \dots, m$.

When $m = n - 1$, it is the definition. When $m = 0$, it becomes this lemma. Now we try to show it is true for $m = k - 1$.

For every $x_1 < \cdots < x_{n-1}$ in $D$, there is a nontrivial $\mu \in M$ satisfying (7), (8) and $\mu(\{x_i\}) = 0$, $i = 1, \dots, k-1$. We will try to find a nontrivial $\nu \in M$ satisfying (7), (8), and $\nu(\{x_i\}) = 0$, $i = 1, \dots, k$. Then, by the induction hypothesis, $M$ is a $WT$ measure vector space and the theorem is proved.

If $\mu_{x_k} \in M$, then we can choose $\nu = \mu$ and we are done.

Otherwise, $\mu(\{x_k\}) \neq 0$. First, assume $(-1)^k \mu(\{x_k\}) > 0$ and $x_k \neq \pm 1$. Let $x_\varepsilon = x_k - \varepsilon$. Then there is a nontrivial $\mu_\varepsilon \in M$ satisfying (7), (8) with replacing $x_k$ by $x_\varepsilon$, and $\mu_\varepsilon(\{x_i\}) = 0$, $i = 1, \dots, k-1$. Then $(-1)^k \mu_\varepsilon(\{(x_k\}) \leq 0$ for sufficiently small $\varepsilon$ and we can require $\|\mu_\varepsilon\| = 1$. By the compactness argument, there is a sequence $\{\mu_{\varepsilon_i}\}$ that converges to $\hat{\mu}$ as $\varepsilon_i \to 0$. This $\hat{\mu}$ satisfies (7), (8), $\hat{\mu}(\{x_i\}) = 0$, $i = 1, \dots, k-1$, and $(-1)^k \hat{\mu}(\{x_k\}) \leq 0$. If $(-1)^k \hat{\mu}(\{x_k\}) = 0$ then we are done. Otherwise let $\nu = \hat{\mu}(\{x_k\})\mu + \mu(\{x_k\})\hat{\mu} \in M$. $\nu$ satisfies (7), (8), and $\nu(\{x_i\}) = 0$, $i = 1, \dots, k$. Also, $\nu \neq 0$ because if $\nu = 0$, then $\mu_{x_k} = (1/\mu(\{x_k\}))\mu \in M$.

For the cases $(-1)^k \mu(\{x_k\}) < 0$ and $x_k = \pm 1$ the proof is similar.

The set of all Borel signed measures on $[-1, 1]$ can be considered as the dual space of $C[-1, 1]$.

**Lemma 4.3.** *Let $C^*$ denote the dual space of $C[-1, 1]$, the set of all Borel signed measures, and let $x_1 < \cdots < x_{n-1}$ be $n-1$ fixed points in $(-1, 1)$. Also* *denote* $x_0 = -1$ *and* $x_n = 1$. *For* $\varepsilon > 0$ *with* $\varepsilon < \min\{x_i - x_{i-1},\ i = 1, \dots, n\}$ *denote*

$$U_\varepsilon = \bigcup_{i=1}^{n-1} (x_i - \varepsilon, x_i + \varepsilon)$$

*Then,*

$$L_\varepsilon = \{\mu \in C^* | (-1)^i \mu|_{[x_{i-1}, x_i]} \geq 0,\ i = 1, \dots, n,\ \|\mu\| = 1,\ \mu|_{U_\varepsilon} = 0\} \tag{9}$$

*is a compact convex set in the weak*-topology.*

*Proof*. It is obvious that $L_\varepsilon$ is convex. To show it is compact in the weak*-topology it is enough to show $L_\varepsilon$ is closed. Let $\nu \notin L_\varepsilon$. Then, there are three possibilities.

Case 1. There exist some $i$ and a set $E \subset [x_{i-1}, x_i] \backslash U_\varepsilon$ with $(-1)^i \nu|_E < 0$. By the regularity of $\nu$ we can choose $E$ to be compact and an open set $A$ (relative to $[-1, 1]$) with $E \subset A \subset [x_{i-1}, x_i] \backslash U_\varepsilon$ and $|\nu|(A - E) < (1/2)|\nu|(E)$. Now we can find an $f \in C[-1, 1]$ with $\|f\| = 1$, $f = (-1)^i$ on $E$, $f = 0$ on $A^c$, and $(-1)^i \operatorname{sgn}(f) \geq 0$. This implies

$$\int f\, d\nu = \int_E f\, d\nu + \int_{A-E} f\, d\nu \leq (-1)^i \nu(E) + |\nu|(A - E) < 0.$$

On the other hand, for any $\mu \in L_\varepsilon$

$$\int f\, d\mu = \int_A f\, d\mu \geq 0$$

and hence $\nu \notin \overline{L}_\varepsilon$, the closure of $L_\varepsilon$.

Case 2. There exists some $i$ such that $\nu|_{(x_i - \varepsilon, x_i + \varepsilon)} \neq 0$. By similar argument as in the Case 1, $\nu \notin \overline{L}_\varepsilon$.

Case 3. $\nu = c\mu$ for some $\mu \in L_\varepsilon$ and $c \neq 1$. Let

$$f(x) = \begin{cases} (-1)^i & x \in [x_{i-1}, x_i] \backslash U_\varepsilon, \\ \text{linear} & x \in U_\varepsilon. \end{cases}$$

Then, $\int f\, d\nu = c$, but $\int f\, d\mu = 1$ for any $\mu \in L_\varepsilon$. Thus, $\nu \notin \overline{L}_\varepsilon$.

This proves $L_\varepsilon = \overline{L}_\varepsilon$.

*Proof of Theorem* 3.3. Suppose $M$ is not a $WT$ measure vector space. Then by Lemma 4.2 there exist $x_1 < x_2 < \cdots < x_{n-1}$ in $[-1, 1]$ such that no nontrivial $\mu \in M$ satisfies (7) and (8). Assume $x_1 > -1$ and $x_{n-1} < 1$, otherwise we can remove $x_1$ and (or) $x_{n-1}$. Let $L_\varepsilon$, $U_\varepsilon$, and $C^*$ be defined as in Lemma 4.3 with these $x_1, \dots, x_{n-1}$. We always consider $C^*$ as a topological vector space in the weak*-topology. Denote $U = \{x_1, \dots, x_{n-1}\}$, $x_0 = -\infty$, $x_n = \infty$ and let

$$M_0 = \{\mu \mid \mu|_U = 0,\ \mu|_{[-1, 1] \backslash U} = \nu|_{[-1, 1] \backslash U} \text{ for some } \nu \in M\}$$

and

$$M_\varepsilon = \{\mu \mid \mu|_{U_\varepsilon} = 0,\ \mu|_{[-1, 1] \backslash U_\varepsilon} = \nu|_{[-1, 1] \backslash U_\varepsilon} \text{ for some } \nu \in M\}.$$

By Lemma 4.2 $M_0 \cap L_\varepsilon = \varnothing$. Since $M$ is a finite dimensional space, by the compactness argument there exists $\varepsilon_0 > 0$ such that $\dim(M_\varepsilon) = \dim(M_0) = m \leq n = \dim(M)$ and $M_\varepsilon \cap L_\varepsilon = \varnothing$ for all $0 < \varepsilon \leq \varepsilon_0$. So there is a continuous linear functional $F$ on $C^*$ (see [2, 3]) such that

$$F(\mu) = 0 \quad \text{for all } \mu \in M_\varepsilon$$

and

$$F(\mu) > 0 \quad \text{for all } \mu \in L_\varepsilon.$$

In the weak *-topology a continuous linear functional on $C^*$ can be represented by a function of $C[-1, 1]$, i.e.,

$$F(\mu) = \int f_\varepsilon \, d\mu, \qquad \mu \in C^*$$

for some $f_\varepsilon \in C[-1, 1]$. Thus, we have

$$\int f_\varepsilon \, d\mu = F(\mu) = 0, \qquad \text{for all } \mu \in M_\varepsilon \tag{10}$$

and

$$\int f_\varepsilon \, d\mu = F(\mu) > 0, \qquad \text{for all } \mu \in L_\varepsilon. \tag{11}$$

Now (11) and the definition of $L_\varepsilon$ imply

$$(-1)^i f_\varepsilon(x) > 0 \quad \text{for } x \in (x_{i-1}, x_i) \backslash U_\varepsilon, \qquad i = 1, \dots, n.$$

For $\delta > 0$ with $\varepsilon + \delta < \min\{x_i - x_{i-1}, \ i = 1, \dots, n\}$ define

$$f_{\varepsilon,\delta} = \begin{cases} 0 & x \in U_\varepsilon, \\ f_\varepsilon(x) & x \in [-1, 1] \backslash U_{\varepsilon+\delta}, \\ \text{linear} & x \in U_{\varepsilon+\delta} \backslash U_\varepsilon. \end{cases}$$

Set

$$s_j(\varepsilon, \delta) = \int f_{\varepsilon,\delta} \, d\mu_j, \qquad j = 1, \dots, m, \tag{12}$$

where $M_0 = \operatorname{span}\{\mu_1, \dots, \mu_m\}$. Then, if $\mu(\{x_i + \varepsilon\}) = \mu(\{x_i - \varepsilon\}) = 0$ for all $\mu \in M_0$ and $i = 1, \dots, n-1$, we have

$$\lim_{\delta \to 0} s_j(\varepsilon, \delta) = 0, \qquad j = 1, \dots, m.$$

It is easy to see that we can find $h_i \in C[-1, 1]$, $i = 1, \dots, m$, such that $h_i(x) = 0$ for $x \in U_{\varepsilon_0}$ and

$$\int h_i \, d\mu_j = \delta_{ij}, \qquad i, j = 1, \dots, m. \tag{13}$$

Now choose $\varepsilon_k \to 0$ with $0 < \varepsilon_k < (\varepsilon_0/2)$ and $\mu(\{x_i + \varepsilon_k\}) = \mu(\{x_i - \varepsilon_k\}) = 0$. Choose $\delta_k \to 0$ $(\delta_k + \varepsilon_k < \varepsilon_0)$ small enough such that

$$\left\| \sum_{i=1}^m s_i(\varepsilon_k, \delta_k) h_i \right\| < \min_{[-1,1]\backslash U_{\varepsilon_0}} \{|f_{\varepsilon_k}(x)|\} > 0.$$

Let

$$g_k = f_{\varepsilon_k, \delta_k} - \sum_{i=1}^m s_i(\varepsilon_k, \delta_k) h_i.$$

Then, we have, $(-1)^i g_k(x) \geq 0$ for $x \in (x_{i-1}, x_i)$, $(-1)^i g_k(x) > 0$ for $x \in (x_{i-1}, x_i) \backslash U_{\varepsilon_k + \delta_k}$, $i = 1, \dots, n$, and, by (12) and (13), $\int g_k \, d\mu = 0$ for all $\mu \in M_0$. Let

$$g = \sum_{k=1}^\infty \frac{1}{2^k \|g_k\|} g_k \in C[-1, 1].$$

Then, $(-1)^i g(x) > 0$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, n$, and $\int g\,d\mu = 0$ for all $\mu \in M_0$. This and the continuity of $g$ imply that $g$ has $n-1$, and only $n-1$, zeros at $x_1, \ldots, x_{n-1}$. Also, by the definition of $M_0$ we have

$$\int g\,d\mu = 0 \qquad \text{for all } \mu \in M.$$

This means $Pg = 0$. By Theorem 3.1 this contradicts that $Pg$ interpolates $g$ at $n$ or more points.

($\Leftarrow$) We again use Theorem 3.1. Assume $M$ is a $WT$ measure vector space and suppose for some $f \in C[-1, 1]$ with $Pf = 0$, $f$ has less than $n$ zeros including multiplicity. Let $x_1, \ldots, x_k, x_{k+1}, \ldots, x_m$ in $[-1, 1]$ be all the zeros of $f$. Suppose $f$ changes sign at $x_i$, $i = 1, \ldots, k$, and does not change sign at $x_i$, $i = k+1, \ldots, m$. Thus $k + 2(m-k) \le n-1$. By Lemma 4.1 there is a nontrivial $\mu = \sum c_i \mu_i \in M$ satisfying (6) and hence

$$\int f\,d\mu > 0.$$

On the other hand, since

$$Pf = \sum_{i=1}^{n} \left( \int f\,d\mu_i \right) v_i = 0$$

and $v_i$, $i = 1, \ldots, n$, are linearly independent,

$$\sum_{i=1}^{n} \left( \int f\,d\mu_i \right) = 0, \qquad i = 1, \ldots, n,$$

and hence

$$\int f\,d\mu = \sum_{i=1}^{n} c_i \int f\,d\mu_i = 0.$$

This contradiction proves our theorem.

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