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CURVATURE PINCHING FOR THREE-DIMENSIONAL MINIMAL SUBMANIFOLDS IN A SPHERE

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ABSTRACT. In this paper, some pinching theorems for the Ricci curvature and the scalar curvature of three-dimensional compact minimal submanifolds in a sphere are given.

1. INTRODUCTION

Let M^n be an n-dimensional compact orientable minimal submanifold in a unit $(n+p)$-sphere S^{n+p}. In [2] it was proved that if $n \geq 4$ and the Ricci curvature of M^n is larger than $n-2$, then M^n is totally geodesic in S^{n+p}. Recently, the corresponding problem for the three-dimensional case was treated in [4]. The aim of this paper is to improve the result of [4] so that the theorem of [2] is valid for the case $n = 3$. Precisely, we prove

Theorem 1. *Let M^3 be a three-dimensional compact minimal submanifold in a unit sphere S^{3+p}. If the Ricci curvature of M^3 is larger than* 1 *then M^3 is totally geodesic in S^{3+p}.*

Moreover, for lower codimension, we have

Theorem 2. *Let M^3 be a compact orientable minimal submanifold in S^{3+p} with $p \leq 2$. If the Ricci curvature of M^3 is not less than $(5p-4)/(4p-2)$ then M^3 is totally geodesic in S^{3+p}.*

In the same way as in the proof of Theorem 1, we also obtain

Theorem 3. *Let M^3 be a compact minimal submanifold in S^{3+p}. If the scalar curvature of M^3 is larger than* 4 *then M^3 is totally geodesic.*

Throughout this paper, all the manifolds dealt with are smooth and connected.

2. PRELIMINARIES

In this section we state some notations and basic formulas. More details can be found in [4]. Let M^3 be a three-dimensional compact Riemannian manifold

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that is minimally immersed in a unit $(3+p)$-sphere S^{3+p}. We choose a local field of orthonormal frames $e_1, \dots, e_{3+p}$ in S^{3+p} such that, restricted to M^3, the vectors e_1, e_2, and e_3 are tangent to M^3. Unless otherwise stated, we agree on the following ranges of indices: $1 \le i, j, k, \dots \le 3$; $4 \le \alpha, \beta, \dots \le 3+p$. The second fundamental form of M^3 in S^{3+p} is

$$\sigma = \sum_{\alpha, i, j} h_{ij}^{\alpha} \omega^i \otimes \omega^j \otimes e_\alpha, \tag{2.1}$$

of which the length square is $\|\sigma\|^2 = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2$.

Let $UM \to M^3$ be the unit tangent bundle over M^3. We define a function $f\colon UM \to \mathbf{R}$ by

$$f(u) = \|\sigma(u, u)\|^2 = \sum_\alpha \left(\sum_{i,j} h_{ij}^\alpha u^i u^j \right)^2 \tag{2.2}$$

for $u = \sum_i u^i e_i \in UM$. Since UM is compact, f attains its maximum at a vector in UM. Suppose that this vector is $v \in UM_{x_0}$ for some point $x_0 \in M^3$. By taking $e_1 = v$ at x_0 and letting

$$b_{ij} = \sum_\alpha h_{11}^\alpha h_{ij}^\alpha, \tag{2.3}$$

from the maximality of f we can choose vectors e_2 and e_3 at x_0 such that (cf. [4])

$$f(v) = b_{11} = \max_{u \in UM} \{\|\sigma(u, u)\|^2\}, \tag{2.4}$$

$$b_{ij} = 0 \qquad (i \ne j), \tag{2.5}$$

$$2\sum_\alpha (h_{1k}^\alpha)^2 + b_{kk} - b_{11} \le 0 \qquad (k \ne 1), \tag{2.6}$$

$$\sum_\alpha (h_{11i}^\alpha)^2 + \sum_\alpha h_{11}^\alpha h_{11ii}^\alpha \le 0 \tag{2.7}$$

at the point x_0.

The Gauss equation of M^3 is

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.8}$$

from which and the minimality it follows that

$$R_{ij} = 2\delta_{ij} - \sum_{\alpha, k} h_{ik}^\alpha h_{jk}^\alpha \tag{2.9}$$

and

$$R = 6 - \|\sigma\|^2, \tag{2.10}$$

where R_{ijkl}, R_{ij}, and R denote the curvature tensor, the Ricci tensor, and the scalar curvature of M^3, respectively.

Summing up for i in (2.7) and using (2.5) and the Ricci identity, we easily get [4]

$$0 \ge 3f(v) + 2\sum_{\alpha, k \ne 1} b_{kk} (h_{1k}^\alpha)^2 - 2f(v) \sum_{\alpha, k\ne 1} (h_{1k}^\alpha)^2 - \sum_{k \ne 1} (b_{kk})^2 - f(v) b_{11} \tag{2.11}$$

at the point x_0.

Finally, as is well known, the curvature tensor of a three-dimensional manifold can be expressed as

$$R_{ijkl} = \delta_{ik}R_{jl} - \delta_{il}R_{jk} + \delta_{jl}R_{ik} - \delta_{jk}R_{il} - \tfrac{1}{2}R(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.12}$$

3. Proofs of Theorems 1 and 3

We restrict ourselves to the point x_0 where the function f defined by (2.2) attains its maximum. Then, from (2.3) and (2.4) one can easily see that

$$(b_{kk})^2 \le \left(\sum_\alpha (h_{11}^\alpha)^2\right)\left(\sum_\alpha (h_{kk}^\alpha)^2\right) \le (b_{11})^2 \tag{3.1}$$

for $k \neq 1$, from which and the three-dimensional minimality it follows that

$$b_{22} \le 0, \qquad b_{33} \le 0, \qquad \sum_{k\neq 1}(b_{kk})^2 \le \left(\sum_{k\neq 1} b_{kk}\right)^2 = (b_{11})^2. \tag{3.2}$$

From (2.3) and (2.9) we have

$$-\sum_{\alpha, k\neq 1}(h_{1k}^\alpha)^2 = R_{11} - 2 + b_{11}. \tag{3.3}$$

Substituting (3.3) into (2.11) and using (3.2), one can obtain

(3.4)
$$\begin{aligned} 0 &\ge 3f(v) + 2\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^\alpha)^2 + 2f(v)(R_{11} - 2 + b_{11}) - \sum_{k\neq 1}(b_{kk})^2 - f(v)b_{11} \\ &= -f(v) + 2\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^\alpha)^2 + 2f(v)R_{11} + f(v)b_{11} - \sum_{k\neq 1}(b_{kk})^2 \\ &\ge -f(v) + 2\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^\alpha)^2 + 2f(v)R_{11}. \end{aligned}$$

On the other hand, by (3.2), (2.6), and (3.1), we have respectively

$$\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^\alpha)^2 \ge \frac{1}{2}\sum_{k\neq 1} b_{kk}(b_{11} - b_{kk}) = -\frac{1}{2}\sum_i (b_{ii})^2 \tag{3.5}$$

and

$$\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^\alpha)^2 \ge -f(v)\sum_{\alpha, k\neq 1}(h_{1k}^\alpha)^2 = f(v)(R_{11} - 2 + b_{11}). \tag{3.6}$$

Introducing (3.5) and (3.6) into (3.4), we get

$$\begin{aligned} 0 &\ge -f(v) + 2f(v)R_{11} + f(v)(R_{11} - 2) + \frac{1}{2}\left[(b_{11})^2 - \sum_{k\neq 1}(b_{kk})^2\right] \\ &\ge 3f(v)(R_{11} - 1). \end{aligned} \tag{3.7}$$

Thus, if the Ricci curvature of M^3 is larger than 1 then (3.7) implies that $f(v) = 0$, i.e., $\|\sigma\|^2$ vanishes identically. This proves Theorem 1.

In the similar manner, it follows from (2.11), (3.5), (3.6), and (3.1) that

$$\begin{aligned} 0 &\geq 3f(v) + \left(\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 - 2f(v)\sum_{\alpha, k\neq 1}(h_{1k}^{\alpha})^2\right) \\ &\quad + \left(\sum_{\alpha, k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 - \sum_{k\neq 1}(b_{kk})^2 - (b_{11})^2\right) \\ &\geq 3f(v) - 3f(v)\sum_{\alpha, k\neq 1}(h_{2k}^{\alpha})^2 - \frac{3}{2}f(v)b_{11} - \frac{3}{2}\sum_{k\neq 1}(b_{kk})^2 \\ &\geq \frac{3}{2}f(v)\left\{2 - \sum_{\alpha, i}(h_{ii}^{\alpha})^2 - 2\sum_{\alpha, k\neq 1}(h_{2k}^{\alpha})^2\right\} \\ &\geq \frac{3}{2}f(v)\{2 - \|\sigma\|^2(x_0)\}. \end{aligned} \tag{3.8}$$

Thus, if the scalar curvature of M^3 is larger than 4, i.e., $\|\sigma\|^2 < 2$, then (3.8) implies that $f(v) = 0$, i.e., M^3 is totally geodesic. Theorem 3 is proved.

4. Proof of Theorem 2

For a compact orientable minimal submanifold M^3 in S^{3+p}, a standard calculation gives (cf. [4, Lemma 1.2])

$$\int_{M^3}\left\{2\sum_{\alpha, i, j, k, l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{il}^{\alpha}R_{lkjk}) + \frac{1}{p}\|\sigma\|^4 - 3\|\sigma\|^2\right\}{}^*1 \leq 0, \tag{4.1}$$

where ^{*1} denotes the volume element of M^3.

Let $Q(x)$ be the function assigns to each point x of M^3 the minimum of the Ricci curvatures of M^3 at that point x. For each α, let α_i be the eigenvalues of the matrix (h_{ij}^{α}). Then, by (2.12) we have

$$\begin{aligned} \sum_{i, j, k, l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{il}^{\alpha}R_{lkjk}) &= \sum_{i\neq j}(\alpha_i^2 - \alpha_i\alpha_j)(R_{ii} + R_{jj} - \tfrac{1}{2}R) \\ &= 3\sum_i \alpha_i^2 R_{ii} - \tfrac{1}{2}R\sum_i \alpha_i^2 \geq (3Q - \tfrac{1}{2}R)\sum_{i, j}(h_{ij}^{\alpha})^2, \end{aligned}$$

from which and (4.1) it follows that

$$\int_{M^3}\|\sigma\|^2(6Q - R + \tfrac{1}{p}\|\sigma\|^2 - 3){}^*1 \leq 0,$$

i.e., by (2.10),

$$\int_{M^3}\|\sigma\|^2\left(6Q - 9 + \frac{p+1}{p}\|\sigma\|^2\right){}^*1 \leq 0. \tag{4.2}$$

On the other hand, the well-known Simons inequality [5] for $n = 3$ is

$$\int_{M^3}\|\sigma\|^2\left(\frac{3p}{2p-1} - \|\sigma\|^2\right){}^*1 \leq -\int_{M^3}\|\nabla\sigma\|^{2*}1 \leq 0, \tag{4.3}$$

from which and (4.2) we get

$$\int_{M^3} \|\sigma\|^2 \left(Q - \frac{5p-4}{4p-2}\right) {}^*1 \leq 0. \tag{4.4}$$

Thus, if $Q > (5p-4)/(4p-2)$, then (4.4) implies that $\|\sigma\|^2 = 0$ identically. We now consider the case that $Q = (5p-4)/(4p-2)$. Then, (4.2) becomes

$$\int_{M^3} \|\sigma\|^2 \left(\|\sigma\|^2 - \frac{3p}{2p-1}\right) {}^*1 \leq 0,$$

which together with (4.3) gives that $\nabla\sigma = 0$, and hence, since $\|\sigma\|^2$ is constant, $\|\sigma\|^2 = 0$ or $3p/(2p-1)$. Since the Ricci curvature of M^3 is positive everywhere, M^3 cannot be the Clifford hypersurface. Now, Theorem 2 follows directly from the well-known result of [1] for $n = 3$.

Remark. It is clear that the pinching values given here are not the best possible. In general, for each pair (n, p), there is a best pinching value for minimal M^n in S^{n+p}. Really, in [2] the pinching constant $n-2$ for the Ricci curvature is not sharp for $n \neq 4$ and $p \neq 1$. In [3], it was proved that there exists an isometric minimal immersion of $S^3_{1/8}$ into S^9, where $S^3_{1/8}$ denotes the 3-sphere with constant sectional curvature $1/8$. On the other hand, it is well known that every three-dimensional Einstein manifold is of constant curvature. So, perhaps one can surmise that the best possible pinching value of the Ricci curvature for minimal M^3 in S^{3+p} would be $\frac{1}{4}$. However, we have not demonstrated it.

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