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ON THE RETARDED LIÉNARD EQUATION

BO ZHANG

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ABSTRACT. We consider the equation $x''+f(x)x'+g(x(t-h))=0$ in which f, g are continuous with $f(x)>0$ for $x\in R$, h is a nonnegative constant, and $xg(x)>0$ if $|x|\geq k$ for some $k\geq 0$. Necessary and sufficient conditions are given for boundedness of all solutions and their derivatives. When $k=0$ we give necessary and sufficient conditions for all solutions and their derivatives to converge to zero.

We consider the retarded Liénard Equation

$$x''+f(x)x'+g(x(t-h))=0 \tag{1}$$

in which h is a nonnegative constant and g, f are continuous with $f(x)>0$ for all $x\in R$; it is supposed that exist constants $k\geq 0$ and $N>1$ such that

$$xg(x)>0, \qquad x\left(\int_0^x f(\xi)\,d\xi - Nhg(x)\right)>0 \quad \text{if } |x|\geq k. \tag{2}$$

Equation (1), with or without delay, has attracted considerable attention for more than fifty years, particularly in the theory of asymptotic behavior of solutions (cf. [1–16]). In 1965 Burton [3] obtained the following result:

Theorem B. *Suppose that f and g are continuous with $f(x)>0$, $xg(x)>0$ if $x\neq 0$. Then the zero solution of* (1) *with $h=0$ is globally asymptotically stable if and only if*

$$\int_0^{\pm\infty}[f(x)+|g(x)|]\,dx=\pm\infty. \tag{3}$$

Much effort has gone into the modification of signum conditions on f and g when boundedness of solutions is considered (cf. [4, 6, 12, 13, 14, 15]).

For the retarded equation (1), it is often assumed that

$$|g'(x)|\leq L \text{ for some constant } L>0 \tag{4}$$

and

$$xg(x)>0 \quad (x\neq 0), \qquad f(x)>hL \tag{4*}$$

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in order to obtain the results on boundedness and stability (cf. [2, 9, 16]). It is clear that (2) is a generalization of (4*).

We consider the following questions: Is condition (3) still necessary and sufficient for boundedness and stability of solutions of (1)? For the case $h > 0$, is Theorem B still valid when condition (4) is not satisfied? We have an affirmative answer to these questions.

A system equivalent to (1) is

$$\begin{cases} x' = y \\ y' = -g(x(t-h)) - f(x)y. \end{cases} \tag{5}$$

Notice that condition (2) guarantees the global existence of solutions of system (5) for increasing t. Let R^+, R denote the intervals $[0, +\infty)$ and $(-\infty, +\infty)$ respectively. If x or y is written without its argument then the argument is t. We also denote by $V'(t)$ the upper right-hand derivative of $V(t)$ with respect to t if no confusion occurs.

Theorem 1. *If* (2) *is satisfied, then all solutions of* (5) *are bounded if and only if* (3) *holds.*

Proof. Define $F(x) = \int_0^x f(\xi)\,d\xi$, $G(x) = \int_0^x g(\xi)\,d\xi$, and let $x(t) = x(t, \phi, y_0)$, $y(t) = y(t, \phi, y_0)$ be a solution of (5) with $\phi \in C([-h, 0], R)$, $x(\xi) = \phi(\xi)$ for $\xi \in [-h, 0]$, and $y(0) = y_0$; then $(x(t), y(t))$ exists on R^+. We now define

$$V_0(t) = \left[y(t) + F(x(t)) - \int_{-h}^0 g(x(t+s))\,ds\right]^2 + 2G(x(t)) + N\int_{-h}^0\int_{t+s}^t |g(x(\tau))|^2\,d\tau\,ds + P$$

where $P = 2\sup\{|G(\xi)| : |\xi| \le k\}$; then

$$V_0'(t) \le -2(F(x(t)) - Nhg(x(t)))g(x(t)) - (N-1)h|g(x(t))|^2 - (N-1)\int_{-h}^0 |g(x(t+s))|^2\,ds.$$

Next define

$$V_1(t) = V_0(t) + \frac{1}{2}(N-1)h\int_{-h}^0 |g(x(t+s))|^2\,ds; \tag{6}$$

then

$$\begin{aligned} V_1' \le & -2(F(x(t)) - Nhg(x(t)))g(x(t)) - \frac{1}{2}(N-1)h|g(x(t))|^2 \\ & - (N-1)\int_{-h}^0 |g(x(t+s))|^2\,ds \\ = & -\frac{3}{2}(F(x) - Nhg(x))g(x) - \frac{1}{2N}(N-1)F(x)g(x) \\ & - \frac{1}{2N}(F(x) - Nhg(x))g(x) - (N-1)\int_{-h}^0 |g(x(t+s))|^2\,ds \\ \le & -(N-1)F(x)g(x)/2N \quad \text{if } |x| \ge k. \end{aligned} \tag{7}$$

Now define

$$V_2(t) = |y(t)| + \int_{-h}^{0} |g(x(t+s))|\, ds. \tag{8}$$

We then have

$$V_2'(t) \le -f(x(t))|y(t)| + |g(x(t))|. \tag{9}$$

We will use the following constants

$$f_0 = \min\{f(x) : |x| \le k\}, \qquad g^* = \max\{|g(x)| : |x| \le k\},$$
$$F_0 = \min\{F(k), |F(-k)|\}, \quad F^* = \max\{F(k), |F(-k)|\},$$
$$\gamma = F_0(N-1)/4N.$$

Finally we define

$$V(t) = V_1(t) + \gamma V_2(t); \tag{10}$$

then

$$\begin{aligned} V'(t) \le & -2(F(x(t)) - Nhg(x(t)))g(x(t)) - \frac{1}{2}(N-1)h|g(x(t))|^2 \\ & -(N-1)\int_{-h}^{0} |g(x(t+s))|^2\, ds - \gamma f(x)|y(t)| + \gamma|g(x)|. \end{aligned} \tag{11}$$

Choose $Q > k$ such that $3g^*F^* + 2Nh[g^*]^2 - \gamma f_0(Q-k) < -1$ and consider $|x(t)| + |y(t)| \ge Q$:

If $|x(t)| \ge k$ then

$$\begin{aligned} V'(t) \le & -\frac{1}{2N}(N-1)F(x)g(x) - \gamma f(x)|y(t)| + \gamma|g(x)| \\ \le & -\gamma f(x)|y(t)| - (N-1)F(x)g(x)/4N. \end{aligned}$$

If $|x(t)| < k$ then $|y(t)| \ge Q - k$ and

$$\begin{aligned} V'(t) &\le 2|F(x)||g(x)| + 2Nh|g(x)|^2 + \gamma|g(x)| - \gamma f(x)|y(t)| \\ &\le 2F^*g^* + 2Nh[g^*]^2 + \gamma|g^*| - \gamma f_0(Q-k) < -1. \end{aligned}$$

Thus

$$V'(t) < 0 \quad \text{if } |x(t)| + |y(t)| \ge Q. \tag{12}$$

Notice also that

$$\begin{aligned} V'(t) \le & -2|F(x) - Nhg(x)||g(x)| - (N-1)h|g(x)|^2/4 \\ & -(N-1)\int_{-h}^{0} |g(x(t+s))|^2\, ds + M, \end{aligned}$$

where $M = 4\max\{|F(\xi) - Nhg(\xi)||g(\xi)| : |\xi| \le k\} + (N-1)F_0^2/16hN^2$. If there exists $t_1 > 0$ such that $V(0) < V(t_1)$ and

$$V(t_1) = \max_{0 \le s \le t_1} V(s),$$

then

$$|x(t_1)| + |y(t_1)| < Q$$

and

$$\frac{1}{4}(N-1)h|g(x(t_1))|^2 + (N-1)\int_{t_1-h}^{t_1} |g(x(s))|^2\, ds \le M.$$

By (10) it follows that

$$\begin{aligned}
V(t_1) = &\left[y(t_1) + F(x(t_1)) - \int_{t_1-h}^{t_1} g(x(s))\, ds\right]^2 + 2G(x(t_1)) \\
&+ N\int_{-h}^{0}\int_{t_1+s}^{t_1} |g(x(\nu))|^2\, d\nu\, ds + \frac{1}{2}(N-1)h\int_{t_1-h}^{t_1} |g(x(s))|^2\, ds + P \\
&+ \gamma|y(t_1)| + \gamma\int_{-h}^{0} |g(x(t+s))|\, ds \\
\le\ & 2[y(t_1) + F(x(t_1))]^2 + 2\left(\int_{-h}^{0} g(x(t_1+s))\, ds\right)^2 + 2G(x(t_1)) \\
&+ Nh\int_{t_1-h}^{t_1} |g(x(s))|^2\, ds + \frac{1}{2}(N-1)h\int_{t_1-h}^{t_1} |g(x(s))|^2\, ds + P \\
&+ \gamma|y(t_1)| + \gamma\int_{-h}^{0} |g(x(t_1+s))|^2\, ds + \gamma h \\
\le\ & 4|y(t_1)|^2 + 4|F(x(t_1))|^2 + 2h\int_{-h}^{0} |g(x(t_1+s))|^2\, ds + P \\
&+ 2|G(x(t_1))| + \frac{2hNM}{N-1} + \gamma Q + \frac{\gamma M}{N-1} + \gamma h \\
\le\ & 4|F(x(t_1))|^2 + 2|G(x(t_1))| + P + 4Q^2 + \frac{2hM}{N-1} \\
&+ \frac{2NhM}{N-1} + \gamma Q + \frac{\gamma M}{N-1} + \gamma h.
\end{aligned}$$

Let $\overline{K} = \max\{|F(\xi)|^2 + |G(\xi)| : |\xi| \le Q\}$; then

$$V(t_1) \le 6\overline{K} + 4Q^2 + \frac{(2h + 2Nh + \gamma)M}{N-1} + P + \gamma(Q+h) \overset{\text{def}}{=} B.$$

Hence,

$$V(t) \le V(0) + B \quad \text{for all } t \in R^+.$$

Consequently, we have $|y(t)| \le (V(0)+B)/\gamma$ and

$$\left|\int_0^{x(t)} f(\xi)\, d\xi\right| + \left|\int_0^{x(t)} |g(\xi)|\, d\xi\right| \le B^* \quad \text{for some } B^* > 0.$$

By condition (3), it follows that $|x(t)|$ is bounded on R^+. This proves that (3) is sufficient.

Now we show that (3) is necessary. Suppose that condition (3) fails. To be definite, we assume that

$$\int_0^{+\infty} [f(s) + |g(s)|]\, ds < +\infty.$$

Let $\phi \in C([-h, 0], R)$ and $\phi(\xi) > k$, $\xi \in [-h, 0]$, $x_0 = \phi(0)$. Define

$$\tilde{g} = \max\{g(\phi(\xi)) : \xi \in [-h, 0]\}$$

and

$$y_0 = 2 + h\tilde{g} + \int_{x_0}^{+\infty} [f(\xi) + g(\xi)]\, d\xi.$$

Let $(x(t), y(t))$ be the solution of (5) with $x(\xi) = \phi(\xi)$, $\xi \in [-h, 0]$ and $y(0) = y_0$. We claim that $y(s) > 1$ for all $s \in R^+$.

Now suppose that there exists $t_1 > 0$ such that $y(t_1) = 1$ and $y(s) > 1$ on $[0, t_1)$. Consequently, $x(t)$ is increasing on $[0, t_1)$.

Case 1. Suppose that $t_1 \le h$. Integrate the second equation in (5) from 0 to t_1 to obtain

$$\begin{aligned} y(t_1) &= y_0 - \int_0^{t_1} g(x(s-h))\, ds - \int_0^{t_1} f(x(s))x'(s)\, ds \\ &\ge y_0 - h\tilde{g} - \int_{x(0)}^{x(t_1)} f(\xi)\, d\xi \ge y_0 - h\tilde{g} - \int_{x_0}^{+\infty} f(\xi)\, d\xi > 1, \end{aligned}$$

a contradiction.

Case 2. Suppose that $t_1 > h$. Integrating the second equation in (5) from 0 to t_1, we have

$$\begin{aligned} y(t_1) &= y_0 - \int_0^{h} g(x(s-h))\, ds - \int_h^{t_1} (g(x(s-h))\, ds - \int_0^{t_1} f(x(s))x'(s)\, ds \\ &\ge y_0 - h\tilde{g} - \int_0^{t_1-h} g(x(s))\, ds - \int_0^{t_1} f(x(s))x'(s)\, ds \\ &\ge y_0 - h\tilde{g} - \int_0^{t_1-h} g(x(s))x'(s)\, ds - \int_0^{t_1} f(x(s))x'(s)\, ds \\ &\ge y_0 - h\tilde{g} - \int_{x_0}^{+\infty} g(\xi)\, d\xi - \int_{x_0}^{+\infty} f(\xi)\, d\xi > 1, \end{aligned}$$

a contradiction. Thus $y(t) > 1$ on R^+ and $x(t) \ge t + x_0 \to +\infty$ as $t \to +\infty$. This completes the proof of Theorem 1.

Theorem 2. *If* (2) *is satisfied with* $k = 0$, *then the zero solution of* (5) *is globally asymptotically stable if and only if* (3) *holds.*

Proof. Suppose (2) and (3) hold with $k = 0$. It is clear from the proof of Theorem 1 that the zero solution of (5) is stable. Let $(x(t), y(t))$ be any solution of (5) and $V_1(t)$ be defined in (6), then by (7) we have

$$V_1'(t) \le -(N-1)F(x(t))g(x(t))/2N.$$

This implies that $\lim_{t\to+\infty} \inf |x(t)| = 0$.

Now suppose that $\lim_{t\to+\infty} \sup |x(t)| = \lambda > 0$. Then there exist sequences $\{t_n\}$, $\{t_n'\}$ such that $0 < t_n < t_n'$, $t_n \to +\infty$ as $n \to +\infty$ and

$$|x(t_n)| = \frac{\lambda}{3}, \qquad |x(t_n')| = \frac{2\lambda}{3}, \qquad \left(\frac{\lambda}{3}\right) < |x(t)| < \frac{2\lambda}{3} \quad \text{on } (t_n, t_n').$$

Let

$$\widetilde{F} = \min\left\{F(\xi)g(\xi) | \frac{\lambda}{3} \le |\xi| \le \frac{2\lambda}{3}\right\}.$$

Thus for $t \geq t'_n$, we have

$$\begin{aligned} V_1(t) &\leq V_1(t_1) - (N-1)\int_{t_1}^{t} F(x(s))g(x(s))\,ds/2N \\ &\leq V_1(t_1) - (N-1)\sum_{j=1}^{n}\int_{t_j}^{t'_j} F(x(s))g(x(s))\,ds/2N \\ &\leq V_1(t_1) - (N-1)\widetilde{F}\sum_{j=1}^{n}(t'_j - t_j)/2N. \end{aligned}$$

Since (2) and (3) hold, there exists a constant $B > 0$ such that

$$|x(t)| + |y(t)| \leq B \quad \text{for } t \geq t_0.$$

Hence,

$$\frac{\lambda}{3} = \frac{2\lambda}{3} - \frac{\lambda}{3} \leq \int_{t_j}^{t'_j} |x'(s)|\,ds \leq B(t'_j - t_j)$$

and

$$\lambda \leq 3B(t'_j - t_j) \quad \text{for } j = 1, 2, \ldots.$$

This yields

$$V_1(t) \leq V_1(t_1) - \frac{(N-1)\widetilde{F}n\lambda}{6NB} \to -\infty \quad \text{as } n \to +\infty,$$

a contradiction. Therefore, $\lim_{t\to+\infty}|x(t)| = 0$.

By (6) and (7), it follows that $V_1(t) \to C \geq 0$ as $t \to +\infty$ for some constant C. Since $x(t) \to 0$ as $t \to +\infty$, we have $y^2(t) \to C$ as $t \to +\infty$. Now suppose $C > 0$. Then there exists a constant $T > 0$ such that $t \geq T$ implies $y^2(t) \geq C/2$. Without loss of generality, we may assume that $y(t) \geq (C/2)^{1/2}$ for $t \geq T$. Thus $x'(t) \geq (C/2)^{1/2}$ for $t \geq T$ so that $x(t) \geq x(T) + (C/2)^{1/2}(t-T) \to +\infty$ as $t \to +\infty$, a contradiction. This yields $\lim_{t\to+\infty} y(t) = 0$. The proof of Theorem 1 shows that condition (3) is necessary for the global asymptotic stability of zero solution of (5). This completes the proof of Theorem 2.

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Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901-4408