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A CONNECTEDNESS PROPERTY OF MAXIMAL MONOTONE OPERATORS AND ITS APPLICATION TO APPROXIMATION THEORY

LIBOR VESELÝ

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ABSTRACT. We prove a connectedness property of a general maximal monotone operator on a Hilbert space. As a consequence we easily obtain the characterization of components of the set of discontinuity points for metric projections of closed sets in Hilbert spaces. We show that these components are pathwise connected, too.

1. INTRODUCTION

The present paper has been inspired by a paper of Westphal and Frerking [6], in which the classical theory of monotone operators became a powerful and elegant tool for the investigation of the set of points of discontinuity of metric projections onto arbitrary closed subsets of Hilbert spaces. We prove that the image of certain connected sets by a maximal monotone operator is connected (Theorem 1). As an easy consequence of this general theorem we get a version of a result of Balaganskiĭ [1], which provides the description of components (or, equivalently, pathwise connected components) of the set of those points, at which the metric projection to a closed set in Hilbert space is not continuous (Theorem 2). As a corollary we obtain the result from [6].

Throughout this paper, X will denote a real Hilbert space. The open ball, closed ball, and sphere with center $x \in X$ and radius $r \geq 0$ will be denoted by $B(x; r)$, $\overline{B}(x; r)$, and $S(x, r)$ ($B(x; 0) = \varnothing$, $\overline{B}(x; 0) = S(x; 0) = \{x\}$). The closed convex hull of a set $A \subset X$ will be denoted by $\overline{\text{co}}\,A$. By I we shall denote the identity mapping of X; $[x, y] := \{(1-t)x + ty;\ 0 \leq t \leq 1\}$ for $x, y \in X$.

For a multivalued mapping $T: X \to 2^X$, the graph, range, and domain of T (i.e., the set of x with $T(x) \neq \varnothing$) will be denoted by $G(T)$, $R(T)$, and $D(T)$. By T/S we denote the restriction of T on a set S, i.e.,

$$wG(T/S) = G(T) \cap (S \times X).$$

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We shall sometimes identify a point y with the singleton $\{y\}$; hence $y \in T(x)$ and $T(x) = y$ will mean the same.

2. Main result

Let us recall the basic facts from the theory of monotone operators. We refer the reader to a classical book by Brézis [2], or to [5].

$T : X \to 2^X$ is a monotone operator if $\langle x-y, x^*-y^*\rangle \geq 0$ whenever (x, x^*) and (y, y^*) are elements of $G(T)$. A monotone operator T is maximal if the set $G(T)$ is a maximal element (w.r.t. inclusion of sets) of the family of all graphs of monotone operators on X.

If T is a maximal monotone operator on X and $\lambda > 0$, then $\overline{D(T)}$ is a convex set and any element z of X has a unique representation of the form

$$z = J_\lambda(z) + \lambda T_\lambda(z) \text{ with } (J_\lambda(z), T_\lambda(z)) \in G(T).$$

Moreover, the mappings J_λ and T_λ are Lipschitz with constants 1 and λ^{-1}, respectively. If in addition $z \in D(T)$, then

$$||J_\lambda(z) - z|| \leq \lambda \cdot \min\{||z^*||; z^* \in T(z)\}.$$

An important example of a maximal monotone operator represents the subdifferential $T = \partial f : X \to 2^X$ of a continuous convex function $f : X \to R$, defined by $\partial f(x) = \{x^* \in X; f(y) - f(x) \geq \langle y - x, x^*\rangle \text{ for every } y \in X\}$. The Hahn-Banach theorem implies $D(\partial f) = X$.

Theorem 1. *Let T be a maximal monotone operator on a real Hilbert space X. If S is a connected and relatively open subset of $\overline{D(T)}$, then the sets $G(T/S)$ and $T(S)$ are pathwise connected.*

Proof. Let (x, x^*) and (y, y^*) be two points of $G(T/S)$. The convexity of $\overline{D(T)}$ implies that S is pathwise connected [3, 6.3.10(a)]. Hence there exists a continuous p: $[0, 1] \to S$ with $p(0) = x$ and $p(1) = y$. Denote $P = p([0, 1])$. The compactness of P implies that $\operatorname{dist}(P, \overline{D(T)})\backslash S) = d > 0$ and there exists a finite set $\{x_1, \cdots, x_n\} \subset D(T)$ such that $\operatorname{dist}(z, \{x_1, \cdots, x_n\}) < d/3$ whenever $z \in P$. Choose arbitrarily $x_i^* \in T(x_i)$ for $i = 1, \cdots, n$. Denote

$$M = \max(\{||x_i^*||; 1 \leq i \leq n\} \cup \{||x^*||, ||y^*||\}), \quad z_x = x + \lambda x^*, \quad z_y = y + \lambda y^*$$

where $\lambda > 0$ is chosen in such a way that $\lambda M < d/3$.

Let us define a continuous mapping $q : [-1, 2] \to X$ by $q/_{[0,1]} = p$, $q(-1) = z_x$, $q(2) = z_y$, q affine on $[-1, 0]$ and on $[1, 2]$. Denote $Q = q([-1, 2]) = [z_x, x] \cup P \cup [y, z_y]$.

Let z be an arbitrary element of Q. If $z \in P$, there exists an index i such that $||z - x_i|| < d/3$, and then $||J_\lambda(z) - z|| \leq ||J_\lambda(z) - J_\lambda(x_i)|| + ||J_\lambda(x_i) - x_i|| + ||x_i - z|| \leq 2||z - x_i|| + \lambda||x_i^*|| < d$. If $z \in [z_x, x]$, then $||J_\lambda(z) - x|| \leq ||J_\lambda(z) - J_\lambda(x)|| + ||J_\lambda(x) - x|| \leq ||z - x|| + \lambda||x^*|| \leq ||z_x - x|| + \lambda||x^*|| < 2d/3 < d$. Similarly, $||J_\lambda(z) - y|| < d$ for any $z \in [y, z_y]$. Consequently, $\operatorname{dist}(J_\lambda(z), P) < d$ whenever $z \in Q$. This fact, together with $R(J_\lambda) \subset D(T)$, implies $J_\lambda(z) \in S$ for any $z \in Q$. Consequently, $J_\lambda \circ q$ is a path in S with the initial point $J_\lambda(z_x) = x$ and the terminal point $J_\lambda(z_y) = y$. Moreover, the mapping $t \mapsto (J_\lambda(q(t)), T_\lambda(q(t)))$, $-1 \leq t \leq 2$, is a path in $G(T/S)$ connecting (x, x^*) and (y, y^*). $\square$

3. Connectedness structure of the discontinuity set for metric projections

Let F be a closed subset of X and $x \in X$. We shall use the following notation.

$d_F(x) = \operatorname{dist}(x, F)$ (distance function of F),
$P_F(x) = F \cap \overline{B}(x; d_F(x))$ (metric projection of F),
$f_F(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}d_F^2(x)$,
$C_F = \{x \in X; P_F(x)$ is a singleton and P_F is u.s.c. at $x\}$ ("continuity points" of P_F).

For the following properties, the reader is referred to [6] and the references therein.

The metric projection P_F is always monotone and the function f_F is convex and continuous on X. Moreover, $G(P_F) \subset G(\partial f_F)$. The subdifferential ∂f_F and the set C_F have the following alternative representations.

$$\partial f_F(x) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}(F \cap \overline{B}(x; d_F(x) + \varepsilon)), \tag{1}$$

$$C_F = \{x \in X; P_F(x) = \partial f_F(x)\}. \tag{2}$$

It follows from (1) that

$$\partial f_F(x) \subset \overline{B}(x; d_F(x)) \cap \overline{\operatorname{co}}\, F. \tag{3}$$

It is easy to see that the $F \subset C_F$, because $P_F(x) = \partial f_F(x) = \{x\}$ for $x \in F$.
(4) If $K \subset X$ is convex and closed, then the metric projection P_K is single-valued and continuous on X (even nonexpansive, see, e.g., [4]).

Remark. The definition of C_F implies that it is exactly the set of points of Fréchet differentiably of f_F, and therefore, since X is an Asplund space, it is a dense G_δ-subset of X (cf. [4]).

Lemma 1. $\partial f_F(x) \cap S(x; d_F(x)) = P_F(x)$ *for each* $x \in X$.

Proof. Only the inclusion $\partial f_F(x) \cap S(x; d_F(x)) \subset P_F(x)$ for $x \notin F$ is not obvious. Without any loss of generality we can suppose $x = 0$. Denote $r = d_F(0)$.

Let $z \in S(0; r)\backslash P_F(0)$. There exists a $\delta > 0$ such that $B(z; \delta) \cap F = \varnothing$. Hence, by (1)

$$\partial f_F(0) \subset \bigcap_{\varepsilon>0} \overline{\operatorname{co}}(\overline{B}(0; r+\varepsilon)\backslash B(z; \delta)). \tag{5}$$

Choose an $\varepsilon > 0$ such that $\varepsilon(2r+\varepsilon) \le \delta^2$. Then for any $y \in \overline{B}(0; r+\varepsilon)\backslash B(z; \delta)$ we have

$$\langle z, y\rangle = \tfrac{1}{2}(\|z\|^2 + \|y\|^2 - \|z-y\|^2) \le \tfrac{1}{2}(r^2 + (r+\varepsilon)^2 - \delta^2) \le r^2 - \tfrac{1}{2}\delta^2.$$

Consequently, $\langle z, y\rangle \le r^2 - \delta^2/2$ for any $y \in \overline{\operatorname{co}}(\overline{B}(0; r+\varepsilon)\backslash B(z; \delta))$. But this, together with (5) and $\langle z, z\rangle = r^2$, implies $z \notin \partial f_F(0)$. □

Lemma 2. *Let S be a component of the set $\overline{\operatorname{co}}\, F\backslash F$. If $\partial f_F(x) \cap S \neq \varnothing$ then $P_{\overline{\operatorname{co}} F}(x) \in S$.*

Proof. Obviously $d_{\overline{\text{co}}\,F}(x) \le d_F(x)$. (4) implies $d_{\overline{\text{co}}\,F}(x) < d_F(x)$ and $P_{\overline{\text{co}}\,F}(x) \notin F$ (otherwise $P_{\overline{\text{co}}\,F}(x) = P_F(x) = \partial f_F(x) \subset F$). Consequently

$$\begin{aligned} P_{\overline{\text{co}}\,F}(x) &= \overline{B}(x\,;\, d_{\overline{\text{co}}\,F}(x)) \cap \overline{\text{co}}\,F = \overline{B}(x\,;\, d_{\overline{\text{co}}\,F}(x)) \cap (\overline{\text{co}}\,F \backslash F) \\ &\subset B(x\,;\, d_F(x)) \cap (\overline{\text{co}}\,F \backslash F) = B(x\,;\, d_F(x)) \cap \overline{\text{co}}\,F =: A, \end{aligned}$$

where $A \subset B(x\,;\, d_F(x))$ is a connected (in fact convex) subset of $\overline{\text{co}}\,F\backslash F$. But by (3) and Lemma 1, $\partial f_F(x) \cap S \subset A$. Therefore $A \subset S$. □

The following theorem is due in main to Balaganskiĭ [1], whose proof is rather long and complicated. We obtain this result as an almost immediate consequence of Theorem 1 and the two simple lemmas above.

Theorem 2. *Let F be a closed subset of a real Hilbert space X. Let $\mathcal{S}$ and $\mathcal{M}$ be the families of all components of the set $\overline{\text{co}}\,F\backslash F$ and of the set $X\backslash C_F$, respectively. Then the following holds.*

(a) $X\backslash C_F = \{x \in X\,;\, \partial f_F(x) \cap (\overline{\text{co}}\,F\backslash F) \neq \varnothing\}$.
(b) *The mapping $S \mapsto M_S = \{x \in X\,;\, \partial f_F(x) \cap S \neq \varnothing\}$ is a bijection of $\mathcal{S}$ onto $\mathcal{M}$. Moreover, $M_S = (X\backslash C_F) \cap P^{-1}_{\overline{\text{co}}\,F}(S)$.*
(c) *The inverse of the bijection is the mapping $M \mapsto S_M$ which can be defined in the following way: for $M \in \mathcal{M}$ choose an arbitrary $x \in M$ and denote by S_M the element of $\mathcal{S}$ which contains the point $P_{\overline{\text{co}}\,F}(x)$.*
(d) *The elements of $\mathcal{M}$ and of $\mathcal{S}$ are pathwise connected and relatively open in $X\backslash C_F$ and in $\overline{\text{co}}\,F\backslash F$, respectively.*

Proof. (a) follows easily from (3), Lemma 1, from the convexity of $\partial f_F(x)$ for $x \in X$ and from the fact that X is strictly convex. Each $S \in \mathcal{S}$, being a component of a relatively open subset of a convex set, is pathwise connected and relatively open in $\overline{\text{co}}\,F\backslash F$ [3]. Consider the maximal monotone operator $T = (\partial f_F)^{-1}$. By (3) and by $T/F = I/F$, we have $F \subset R(\partial f_F) = D(T) \subset \overline{\text{co}}\,F$. The convexity of $\overline{D(T)}$ implies $\overline{D(T)} = \overline{\text{co}}\,F$. By Theorem 1, the sets $M_S = T(S)$ are pathwise connected. If $M \in \mathcal{M}$ is such that $M_S \subset M$, then by (a), Lemma 2 and (4), the set $P_{\overline{\text{co}}\,F}(M)$ is connected and it is contained in a component of $\overline{\text{co}}\,F\backslash F$, namely in S. From this we deduce $M = M_S$. This proves (b) and (c). Now, for any $M \in \mathcal{M}$ we have $M = (X\backslash C_F) \cap P^{-1}_{\overline{\text{co}}\,F}(S)$, and hence M is relatively open in $X\backslash C_F$ by (4). □

Corollaries I [6]. *If x is a nonisolated element of $X\backslash C_F$, then there exists a nontrivial path in $X\backslash C_F$ containing x.*

II. *If $X\backslash C_F$ is not an isolated set, then* $\operatorname{card}(X\backslash C_F) \ge \operatorname{card}([0, 1])$.

III. *The set $X\backslash C_F$ has at most* $\operatorname{dens}(X)$ *components* ($\operatorname{dens}(X)$ *is the smallest cardinal number of the form* $\operatorname{card}(A)$ *where A is a dense subset of X*; [3]). *In particular, $X\backslash C_F$ has at most countably many components if X is separable.*

IV. *If $X\backslash F$ is bounded and connected, then the set $X\backslash C_F$ is pathwise connected.*

Proof. I. (d) implies the existence of a point $y \in X\backslash C_F$, distinct from x, such that x and y are elements of the same component of $X\backslash C_F$. Hence x and y can be connected by a path in $X\backslash C_F$.

II follows immediately from I.
III is a consequence of (b).
IV In this case, $\overline{\text{co}}\, F = X$. Apply (b). □

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Via S. Vitale 4, 40125 Bologna, Italy