PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 3, July 1992

GENERALIZATIONS OF CERTAIN NEST ALGEBRA RESULTS

N. K. SPANOUDAKIS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we discuss generalizations from the Hilbert space case to more general settings of certain theorems concerning a nest algebra $\mathscr{L}$, namely, (a) the solvability of $Tx = y$ within $\mathrm{Alg}\mathscr{L}$, (b) the decomposability of finite rank operators of $\mathrm{Alg}\mathscr{L}$ by rank one such operators, and (c) approximability in the strong operator topology of $\mathrm{Alg}\mathscr{L}$ by its finite ranks.

INTRODUCTION

In this paper we discuss generalizations from the Hilbert case to more general settings of certain theorems concerning the class of nest algebras. The techniques are simpler even in the general cases: moreover, some of the theorems are strengthened in other directions too. Motivation for this is the recent trend to investigate well-known and important nest algebra theorems to spaces lacking inner product and projections. Perhaps the best result in this direction is the failure in L^1 of the Larson similarity theorem valid in L^2 (see [1] for discussion in this direction).

Let $\mathscr{L}$ be a subspace lattice (definitions are given below) on a normed space X. The following problem was set by Lance [10]: If $x, y \in X$, find necessary and sufficient conditions such that there exists $T \in \mathrm{Alg}\mathscr{L}$ with $Tx = y$. In case X is a Hilbert space and $\mathscr{L}$ is a complete nest, Lance shows that such a condition is the existence of a constant $K \geq 0$ such that

$$\forall L \in \mathscr{L} \qquad d(y, L) \leq K \cdot d(x, L).$$

(Actually the original phrasing of Lance's theorem is in terms of projections and, therefore, is meaningless in Banach space settings. The formulation we give is clearly an equivalent one, but that is meaningful in the more general case considered.) On the other hand Harrison-Longstaff in [5] show that this condition is not sufficient if $\mathscr{L}$ is finite Boolean algebra and X is a separable Hilbert space. In this paper we show (Theorem 1) that the above condition is necessary and sufficient if X is of the form

$$L^p(E, \mathscr{A}, \mu), \qquad 1 < p \leq +\infty,$$

Received by the editors July 23, 1990.
1991 *Mathematics Subject Classification*. Primary 47A15, 47D30.
Key words and phrases. Nest, nest algebra, finite rank, interpolation, approximation.

where $(E, \mathscr{A}, \mu)$ measure space and $\mathscr{L}$ is a nest of the form described before Theorem 1 below. Thus Theorem 1 is a proper generalization to Lance's theorem because nests of subspaces on a Hilbert space are, by a result of Erdos [3], unitarily equivalent to nests of the form described above, with $p = 2$. Also in the special case of finite nests, our proof is constructive rather than existensial, as in the original version.

Another problem we discuss is whether a finite rank operator R belonging to $\operatorname{Alg}\mathscr{L}$ can be written as a finite sum $R = \sum_{i=1}^{N} R_i$ with each R_i in $\operatorname{Alg}\mathscr{L}$ $(i = 1, \dots, N)$ and of rank one. Ringrose in Erdos paper [4] proves that this is so if X is a Hilbert space and L is a complete nest. Longstaff in [14] proves this if X is a Hilbert space and L is a complete atomic Boolean subspace lattice, and Lambrou [8] extended it to the case of a normed space X. To the contrary Hopenwasser-Moore in [7] constructed a completely distributive $\mathscr{L}$ and a finite rank R in $\operatorname{Alg}\mathscr{L}$ where the decomposition into a sum of rank ones fails. Below (Theorem 2) we improve Ringrose's theorem on nests to the case when the underlying space X is a linear topological space.

A third problem discussed is whether the closure of the set $\{R: R \in \operatorname{Alg}\mathscr{L}$ and rank $R < +\infty\}$ in the strong operator topology is the whole of $\operatorname{Alg}\mathscr{L}$. Erdos in [4] proves that this is so if X is a Hilbert space and L a complete nest. Argyros-Lambrou-Longstaff prove it if X is a normed space and $\mathscr{L}$ an atomic Boolean subspace lattice with just two atoms [2]. Also, if X is a separable Hilbert space and $\mathscr{L}$ a commutative completely distributive lattice this is again true (Laurie-Longstaff [13]). Also Hopenwasser-Laurie-Moore in [6] prove that if L is a commutative subspace lattice then L is completely distributive if and only if the Hilbert-Schmidt (respectively the finite rank) operators in $\operatorname{Alg}\mathscr{L}$ are dense in $\operatorname{Alg}\mathscr{L}$ in any (and hence all) of the strong, ultrastrong, weak or ultraweak operator topologies. In this paper it is proved (Theorem 3) that if X is a normed space and $\mathscr{L}$ a complete nest, then the strong density of the finite rank operators conclusion is valid. Also the operator in $\operatorname{Alg}\mathscr{L}$, which approximates any given operator of $\operatorname{Alg}\mathscr{L}$ at n specified points, can be chosen of rank at most n. This estimate is clearly the optimum one. Note that the original argument Erdos (valid for Hilbert spaces [4]) produces an operator generally of rank $\frac{1}{2}n(n+1)$.

We use the following terminology and notation.

Let X be a (real or complex) linear topological space and $\mathscr{L}$ a set of closed subspaces of X. $\mathscr{L}$ is called *subspace lattice* on X if it contains the trivial subspaces $\{0\}$ and X and if it contains the closed linear span $\bigvee_{i\in I} L_i$ and the set theoretic intersection $\bigcap_{i\in I} L_i$ whenever $L_i \in L$ $(i \in I)$ for some indexing set I. A subspace lattice $\mathscr{L}$ is called *complete nest* if it is totally ordered by inclusion. As we shall not make any specific use of a complete atomic Boolean lattice, a commutative subspace lattice or a completely distributive lattice we do not state definitions but refer the reader to [8, 13]. If $\mathscr{L}$ is a complete nest of subspaces of X and $0 \neq N \in \mathscr{L}$, we define

$$N_- = \bigvee\{L \in \mathscr{L}: L \subset N\}$$

(where $\subset$ denotes proper inclusion). We also define $0_- = 0$. If L is a subset of X and $x \in X$, we define $d(x, L) = \inf\{\|x - y\|: y \in L\}$.

We define also $\langle x, y, \dots, z\rangle$ as the closed linear span of $\{x, y, \dots, z\}$. If $\mathscr{L}$ is a subspace lattice we define $\operatorname{Alg}\mathscr{L} = \{T \in B(X): \forall L \in \mathscr{L},\ T(L) \subseteq$

$L\}$ where $B(X)$ is the set of (bounded linear) operators on X. An operator $T \in B(X)$ is called a finite rank if $\dim T(X)$ is finite and then we set rank $T = \dim T(X)$.

If $p^* \in X^*$ (X^* is the dual space of X) and $q \in X$ are nonzero, we define $p^* \otimes q$ as the rank one operator on X given by $(p^* \otimes q)x = p^*(x) \cdot q$ for every $x \in X$.

Main part

For convenience we state the following lemma, which is a slight generalization with similar proof of Lemma 3.3 of Ringrose [15].

Lemma 1. *Let $\mathscr{L}$ be a complete nest of subspaces of a linear topological space X. Let $0 \neq p^* \in X^*$ and $0 \neq q \in X$. Then $p^* \otimes q \in \operatorname{Alg}\mathscr{L}$ if and only if there is an $N \in \mathscr{L}$ such that $q \in N$ and $p^*(N_-) = 0$.*

We are now in position to state and prove our key lemma.

Lemma 2. *Let*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \textit{and} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

be vectors in $(\mathbb{C}^n, \|\cdot\|_p)$ where $1 \le p < \infty$, and suppose that for some $K \ge 0$,

$$(*) \qquad \sum_{j=i}^{n} |y_j|^p \le K^p \sum_{j=i}^{n} |x_j|^p \qquad 1 \le i \le n.$$

Then there is an upper-triangular $n \times n$ matrix A such that $Ax = y$ and $\|A\|_p \le K$.

Proof. It is clear that there are upper-triangular matrices A such that $Ax = y$. We shall choose one that in addition satisfies $\|A\|_p \le K$. We proceed by induction on n. The result is clearly true for $n = 1$. Suppose that we have it for $n-1$. If $x_n = 0$ then inequalities $(*)$ give $y_n = 0$. By inductive hypothesis there is matrix

$$B = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1,n-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{n-1,n-1} \end{pmatrix} \text{ such that } B \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \text{ and } \|B\|_p \le K.$$

We define

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ & & \alpha_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Clearly $Ax = y$ and also

$$\|Az\|_p^p = \left\| A \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \right\|_p^p = \left\| B \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \right\|_p^p \le K^p \left\| \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \right\|_p^p \le K^p \|z\|_p^p$$

so $\|A\|_p \le K$. If $x_n \neq 0$, the set

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \; e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis of $\mathbb{C}^n$, so if we write $A = (\alpha_{ij})_{i,j}$, the relation $\|A\|_p = \|(\alpha_{ij})_{i,j}\|_p \le K$ is equivalent to

$$\begin{aligned} &\|A(\lambda_n \cdot x + \lambda_1 \cdot e_1 + \cdots + \lambda_{n-1} \cdot e_{n-1}\|_p \\ &\quad \le K\|(\lambda_n \cdot x + \lambda_1 \cdot e_1 + \cdots + \lambda_{n-1} \cdot e_{n-1}\|_p \quad (\text{all } \lambda_1, \ldots, \lambda_n \in \mathbb{C}). \end{aligned}$$

This in turn is equivalent to (if $Ax = y$)

$$\left\| \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \lambda_1 \begin{pmatrix} \alpha_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_{n-1} \begin{pmatrix} \alpha_{1,n-1} \\ \vdots \\ \alpha_{n-1,n-1} \\ 0 \end{pmatrix} \right\|_p \le K \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ 0 \end{pmatrix} \right\|_p$$

and hence to

$$\begin{aligned} &|y_1 + \lambda_1\alpha_{11} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p + |y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}|^p + \cdots \\ (1) \quad &+ |y_{n-1} + \lambda_{n-1}\alpha_{n-1,n-1}|^p + |y_n|^p \\ &\le K^p(|x_1 + \lambda_1|^p + \cdots + |x_{n-1} + \lambda_{n-1}|^p + |x_n|^p) \quad (\text{all } \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}). \end{aligned}$$

We investigate first the case $x_1 \neq 0$. If $K \ge |y_1/x_1|$ then we set $\alpha_{11} = y_1/x_1$, $\alpha_{12} = \cdots = \alpha_{1n} = 0$. By the inductive hypothesis there are $\alpha_{i,j}$ $(i, j = 2, \ldots, n)$ such that

$$\begin{pmatrix} \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and $\|(\alpha_{i,j})_{i,j=2,\ldots,n}\|_p \le K$ (hence (1) holds). Now set

$$A = \begin{pmatrix} y_1/x_1 & 0 & \cdots & & 0 \\ & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & \alpha_{nn} \end{pmatrix}.$$

If $K < |y_1/x_1|$ (and hence $y_1 \neq 0$) then we set $\alpha_{11} = K^p|x_1/y_1|^p(y_1/x_1)$ and we have

$$\begin{aligned} &|y_1 + \lambda_1\alpha_{11} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p \\ &\quad = |K^{p-1}|x_1/y_1|^p(y_1/x_1)K(x_1 + \lambda_1) \\ &\qquad + y_1 - K^p|x_1/y_1|^p y_1 + \lambda_2\alpha_{12} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p. \end{aligned}$$

If $y_2 \neq 0$ then we set

$$\alpha_{1i} = (1 - K^p|x_1/y_1|^p)(y_1/y_2)\alpha_{2i} \qquad i = 2, \ldots, n$$

(the α_{2i} will be determined below) and we have, for $1 < p < +\infty$ by Hölder's inequality,

$$\begin{aligned}
&|y_1 + \lambda_1\alpha_{11} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p \\
&\quad = |(K^{p-1}|x_1/y_1|^p(y_1/x_1))K(x_1+\lambda_1) \\
&\qquad + (1 - K^p|x_1/y_1|^p)^{1/q+1/p}\frac{y_1}{y_2}[y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}]|^p \\
&\quad \le [(K^{p-1}|x_1/y_1|^{p-1})^q + (1 - K^p|x_1/y_1|^p)]^{p/q} \\
&\qquad \cdot [K^p|x_1+\lambda_1|^p + |y_1/y_2|^p(1 - K^p|x_1/y_1|^p)|y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}|^p] \\
&\quad = K^p|x_1+\lambda_1|^p + \frac{1}{|y_2|^p}(|y_1|^p - K^p|x_1|^p)|y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}|^p
\end{aligned}$$

where $1/p + 1/q = 1$.

For the case $p = 1$ we argue similarly but using the triangle instead of the Hölder inequality. Therefore

$$\begin{aligned}
&|y_1 + \lambda_1\alpha_{11} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p + |y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}|^p \\
&\quad \le K^p|x_1+\lambda_1|^p \\
&\qquad + \left[\frac{1}{|y_2|^p}(|y_1|^p - K^p|x_1|^p) + 1\right] \cdot |y_2 + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{2,n-1}|^p.
\end{aligned} \tag{2}$$

If we define $\gamma = \frac{1}{|y_2|}(|y_1|^p - K^p|x_1|^p + |y_2|^p)^{1/p} > 0$ then we have (2) $= K^p|x_1+\lambda_1|^p + |\gamma y_2 + \lambda_2\alpha_{22}\gamma + \cdots + \lambda_{n-1}\alpha_{2,n-1}\gamma|^p$. Finally we define $\alpha'_{2i} = \gamma\alpha_{2i}$ $(i = 2, \ldots, n)$, and we observe that

$$\begin{aligned}
|\gamma y_2|^p + |y_3|^p + \cdots + |y_n|^p &= |y_1|^p - K^p|x_1|^p + |y_2|^p + |y_3|^p + \cdots + |y_n|^p \\
&\le K^p(|x_2|^p + \cdots + |x_n|^p).
\end{aligned}$$

So, by the inductive hypothesis there is a matrix

$$B = \begin{pmatrix} \alpha'_{22} & \alpha'_{23} & \cdots & \alpha'_{2n} \\ & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \alpha_{nn} \end{pmatrix} \quad \text{such that } B \cdot \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \gamma y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

and

$$|\gamma y_2| + \lambda_2\alpha'_{22} + \cdots + \lambda_{n-1}\alpha'_{2,n-1}|^p + \cdots + |y_n|^p \le K^p(|x_2+\lambda_2|^p + \cdots + |x_n|^p).$$

We take $A = (\alpha_{i,j})_{i,j=1,\ldots,n}$ with $\alpha_{i,j} = 0$ for $i > j$ and the other elements are determined above via B. We have

$$\begin{aligned}
\sum_{i=1}^n \alpha_{1i}x_i = \alpha_{11}x_1 + \sum_{i=2}^n \alpha_{1i}x_i &= \alpha_{11}x_1 + (1 - K^p|x_1/y_1|^p)(y_1/y_2) \cdot \sum_{i=2}^n \alpha_{2i}x_i \\
&= \alpha_{11}x_1 + (1 - K^p|x_1/y_1|^p)(y_1/y_2) \cdot 1/\gamma \cdot \sum_{i=2}^n \alpha'_{2i}x_i \\
&= K^p|x_1/y_1|^p y_1 + (1 - K^p|x_1/y_1|^p)(y_1/y_2) \cdot 1/\gamma \cdot \gamma y_2 = y_1,
\end{aligned}$$

and, because of the analogous property of B, we have $Ax = y$.

The way of finding A shows that (1) holds and so $\|A\|_p \le K$.

In case $y_2 = 0$ we have (recall that we have already defined $\alpha_{11} = K^p|x_1/y_1|^p(y_1/x_1)$) that

$$\begin{aligned}&|y_1 + \lambda_1\alpha_{11} + \lambda_2\alpha_{12} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p \\ &\quad = |K^{p-1}|x_1/y_1|^p(y_1/x_1)K(x_1+\lambda_1) + y_1(1 - K^p|x_1/y_1|^p) \\ &\qquad + \lambda_2\alpha_{12} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p.\end{aligned}$$

If we define $\gamma = (1 - K^p|x_1/y_1|^p)^{1/p} > 0$ then we have (because $p = 1 + p/q$)

$$\begin{aligned}&|y_1 + \lambda_1\alpha_{11} + \cdots + \lambda_{n-1}\alpha_{1,n-1}|^p \\ &= \Bigg|K^{p-1}|x_1/y_1|^p(y_1/x_1)K(x_1+\lambda_1) \\ &\quad + \gamma^{p/q}\gamma\left(y_1 + \lambda_2\frac{\alpha_{12}}{\gamma^p} + \lambda_3\frac{\alpha_{13}}{\gamma^p} + \cdots + \lambda_{n-1}\frac{\alpha_{1,n-1}}{\gamma^p}\right)\Bigg|^p \\ &\le [(K^{p-1}|x_1/y_1|^{p-1})^q + \gamma^p]^{p/q} \\ &\quad \cdot\left[K^p|x_1+\lambda_1|^p + \gamma^p\left|y_1 + \lambda_2\frac{\alpha_{12}}{\gamma^p} + \lambda_3\frac{\alpha_{13}}{\gamma^p} + \cdots + \lambda_{n-1}\frac{\alpha_{1,n-1}}{\gamma^p}\right|^p\right] \\ &= K^p|x_1+\lambda_1|^p + |\gamma y_1 + \lambda_2\alpha_{12}\gamma^{1-p} + \lambda_3\alpha_{13}\gamma^{1-p} + \cdots + \lambda_{n-1}\alpha_{1,n-1}\gamma^{1-p}|^p.\end{aligned}$$

We set $\alpha'_{1i} = \alpha_{1i}\gamma^{1-p}$ $i = 2, \ldots, n$.

Since

$$|\gamma y_1|^p + |y_3|^p + \cdots + |y_n|^p = |y_1|^p - K^p|x_1|^p + |y_3|^p + \cdots + |y_n|^p \le K^p(|x_2|^p + \cdots + |x_n|^p)$$

by the inductive hypothesis, there is matrix

$$B = \begin{pmatrix} \alpha'_{12} & \alpha'_{13} & \cdots & \alpha'_{1n} \\ & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \alpha_{nn}\end{pmatrix} \quad \text{such that } B\cdot\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}\gamma y_1 \\ y_3 \\ \vdots \\ y_n\end{pmatrix} \quad \text{and } \|B\|_p \le K.$$

We take $A = (\alpha_{i,j})_{i,j=1,\ldots,n}$ with $\alpha_{22} = \alpha_{23}\cdots = \alpha_{2n} = 0$, $\alpha_{i,j} = 0$ for $i > j$ and the other elements as determined above via B. It is clear that $Ax = y$. So we also have the $\|A\|_p \le K$. If $x_1 = 0$ then we take $\alpha_{11} = 0$, work like the case $K < |y_1/x_1|$, and the proof is complete. □

We remark that the matrix $A = (\alpha_{i,j})$ can be chosen to have also the following property: if for some $i_0 \in \{1, \ldots, n\}$ we have $y_{i_0} = 0$, then $\alpha_{i_0,j} = 0$ $j = 1, \ldots, n$. Below we give the $p = \infty$ version of the previous lemma.

Lemma 3. *Let*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix}$$

be vectors in $(\mathbb{C}^n, \|\ \|_\infty)$, *and suppose that for*

$$1 \le i \le n \ \max\{|y_i|, |y_{i+1}|, \ldots, |y_n|\} \le K\max\{|x_i|, |x_{i+1}|, \ldots, |x_n|\}$$

for some $K \ge 0$. *Then there is an upper-triangular* $n \times n$ *matrix* A *such that* $Ax = y$ *and* $\|A\|_\infty \le K$.

Proof. From the hypotheses, for each $i \in \{1, \ldots, n\}$ there is a $j(i) \in \{i, i+1, \ldots, n\}$ such that $|y_i| \le K|x_{j(i)}|$. If $x_{j(i)} \ne 0$ then we define $\alpha_{i,j(i)} = y_i/x_{j(i)}$

else (in which case $y_i = 0$ as well) we define $\alpha_{i,j(i)} = 0$. We also set $\alpha_{i,j} = 0$ for $j \neq j(i)$ and $i = 1, 2, \ldots, n$. It is clear that $A = (\alpha_{i,j})_{i,j=1,\ldots,n}$ is upper-triangular. Also

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \alpha_{i,j(i)} \cdot x_{j(i)} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ y_i \\ \vdots \end{pmatrix}$$

and

$$\begin{aligned} \|A\|_\infty &= \max\{|\alpha_{i,j(i)}|: i = 1, 2, \ldots, n\} \\ &= \max\{|y_i|/|x_{j(i)}|: i = 1, 2, \ldots, n\} \leq K, \end{aligned}$$

completing the proof. □

Let $(E, \mathscr{A}, \mu)$ be a measure space, and let I be a totally ordered (say by $\leq$) set with a minimum and a maximum element. We suppose that $\{E_i, i \in I\} \subseteq \mathscr{A}$ is such that

(i) If $i, j \in I$ and $i < j$ then $E_i \subset E_j$ a.e.,
(ii) $\mu(E_{\min I}) = 0$, and
(iii) $E_{\max I} = E$ a.e.

If $1 < p \leq +\infty$ then we define $L_i \subseteq L^p$ for $i \in I$ by

$$L_i = \{f \in L^p(E, \mathscr{A}, \mu): f(E - E_i) = 0\}.$$

We define also $\mathscr{L} = \{L_i: i \in I\}$. Then $\mathscr{L}$ is a nest over $L^p(E, \mathscr{A}, \mu)$. Such nests where considered by Larson in [11]. Moreover for Hilbert spaces, Erdos [3] has shown that all nests arise in this way for $p = 2$ and appropriate A. For the nests we consider we show the following interpolation property, which generalizes the result of Lance in [10].

Theorem 1. *Let $\mathscr{L}$ be the nest defined above. The following statements are equivalent*:

(i) $Tx = y$ *for some* $T \in \operatorname{Alg}\mathscr{L}$,
(ii) $K = \sup_{i\in I}(d(y, L_i)/d(x, L_i)) < +\infty$.

If these conditions are satisfied then T can be chosen so that $\|T\| = K$. Also if for some $S \in \operatorname{Alg}\mathscr{L}$ we have $Sx = y$, then $\|S\| \geq K$.

Proof. Let us for the moment assume that I is finite, say $I = \{0, 1, 2, \ldots, n\}$ where $N \in \mathbb{N}$. In this case we shall show that if $d(y, L_i) \leq K\, d(x, L_i)$ $(i = 0, 1, \ldots, n)$ for some $K > 0$ then there is $T \in \operatorname{Alg}\mathscr{L}$ such that $Tx = y$ and $\|T\|_p \leq K$. We shall investigate the case $1 \leq p < +\infty$. (For $p = +\infty$ the proof is similar.) We define $E_i' = E_i - E_{i-1}$ $i = 1, 2, \ldots, n$. Then $\mu(E_i' \cap E_j') = 0$ for $i \neq j$ and $\bigcup_{i=1}^n E_i' = E$ a.e. So for each $z \in L^p(E, \mathscr{A}, \mu)$ we have $z = \sum_{i=1}^n \chi_{E_i'} z$ a.e. where for $A \in \mathscr{A}$ the map $\chi_A: E \to \mathbb{R}$ denotes the characteristic function of A.

We define $z_i = \chi_{E_i'} z$, and so $\|z\|_p^p = \sum_{i=1}^n \|z_i\|_p^p$. Also we have

$$d(z, L_i) = \inf\{\|z-f\|_p : f \in L_i\} = \inf\left\{\left(\int_E |z-f|^p\right)^{1/p} : f \in L_i\right\}$$
$$= \inf\left\{\left(\int_{E_i} |z_1 + z_2 + \cdots + z_i - f|^p + \int_{E-E_i} |z_{i+1} + \cdots + z_n|^p\right)^{1/p} : f \in L_i\right\}$$
$$= \|z_{i+1} + \cdots + z_n\|_p = (\|z_{i+1}\|_p^p + \cdots + \|z_n\|_p^p)^{1/p}.$$

So the conditions $d(y, L_i) \le K\, d(x, L_i)$ $i = 0, 1, \ldots, n-1$ are equivalent to

$$\sum_{j=i}^n \|y_j\|^p \le K^p \sum_{j=i}^n \|x_j\|^p \qquad 1 \le i \le n.$$

Thus by Lemma 1 there is a matrix

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{nn} \end{pmatrix}$$

such that

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{nn} \end{pmatrix} \cdot \begin{pmatrix} \|x_1\| \\ \vdots \\ \|x_n\| \end{pmatrix} = \begin{pmatrix} \|y_1\| \\ \vdots \\ \|y_n\| \end{pmatrix};$$

that is, $\sum_{j=i}^n \alpha_{ij}\|x_j\| = \|y_i\|$ $i = 1, \ldots, n$, and, moreover, $\|A\|_p \le K$. By a corollary to the Hahn-Banach theorem, for each $i \in \{1, \ldots, n\}$, if $x_i \ne 0$ there is an $e_i^* \in L^q$ such that $e_i^*(\{f \in L^p : f(E_i') = 0\}) = 0$, $e_i^*(x_i) = \|x_i\|_p$, and $\|e_i^*\| = 1$ (because $d(x_i, \{f(E_i') = 0\}) = \|x_i\|$). If $x_i = 0$ we take $e_i^* = 0$. We define now $T: L^p \to L^p$, as Lance did in his original proof, by

$$T = \sum_{i=1}^n \sum_{j=i}^n (\alpha_{ij}/\|y_i\|) e_j^* \otimes y_i.$$

(If for some i_0 we have $y_{i_0} = 0$ then also $\alpha_{i_0, i_0} = \alpha_{i_0, i_0+1} = \cdots = \alpha_{i_0, n} = 0$ and for $j = i_0, \ldots, n$ by the expression $\alpha_{i_0, j}/\|y_{i_0}\|$ we mean 0.) It is clear that $T \in \operatorname{Alg}\{L_0, \ldots, L_n\}$. Also

$$Tx = \sum_{i=1}^n \sum_{j=i}^n (\alpha_{ij}/\|y_i\|) e_j^*(x_j) y_i$$
$$= \sum_{i=1}^n \sum_{j=i}^n (\alpha_{ij}/\|y_i\|)\|x_j\| y_i = \sum_{i=1}^n \|y_i\|(1/\|y_i\|) \cdot y_i = y.$$

Finally for each $z \in L^p$ we have

$$\|Tz\|^p = \left\|\sum_{i=1}^{n}\left(\sum_{j=i}^{n}\alpha_{ij}e_j^*(z_j)\right)(y_i/\|y_i\|)\right\|^p = \sum_{i=1}^{n}\left\|\sum_{j=i}^{n}\alpha_{ij}e_j^*(z_j)(y_i/\|y_i\|)\right\|^p$$

$$= \sum_{i=1}^{n}\left|\sum_{j=i}^{n}\alpha_{ij}e_j^*(z_j)\right|^p \leq K^p\left(\sum_{j=1}^{n}|e_j^*(z_j)|^p\right)$$

$$\leq K^p\left(\sum_{j=1}^{n}\|e_j^*\|^p\|z_j\|^p\right) \leq K^p\left(\sum_{j=1}^{n}\|z_j\|^p\right) = K^p\|z\|^p.$$

Based on the preceding conclusions, from here on the proof of the general case is similar to Lance's [10, Theorem 2.3] for Hilbert spaces, and so we omit the steps. (The only difference being the compactness argument used: For a general Banach space X, the unit ball of $B(X^*)$ is compact in following topology: The net $(T_\lambda)_{\lambda\in\Lambda}$ converges to a T if and only if $((T_\lambda f^*)x)_{\lambda\in\Lambda}$ converges to $(Tf^*)x$ for each $f^* \in X^*$ and for each $x \in X$.) □

We now turn to our final two main results, both of which show that the set of rank one operators in a nest algebra is rich enough to determine both all finite rank operators in the nest algebra and, in the strong operator topology, the whole of $\operatorname{Alg}\mathscr{L}$.

Both of these results are known for the Hilbert case, where an elaborate use of Hilbert space techniques is made but are shown here in the context of more general spaces. Generalizations in other directions have been attempted. For example the conclusion of Theorem 2 below is shown in [7] to be false in the case of completely distributive lattices (which is a class of lattices containing nests). It is also shown false in certain finite-distributive lattices [16], a result that is positive if the underlying space is a finite-dimensional Hilbert space [14].

Lemma 4. *Let X be a linear topological space and let $\mathscr{L}$ be a complete nest of subspaces of X. Let also $W \neq 0$ be a finite-dimensional linear subspace of X. Then there is $L_0 \in \mathscr{L}$ such that $W \cap L_0 \neq 0$ and if $W \cap M \neq 0$ for $M \in \mathscr{L}$ then $L_0 \subseteq M$.*

Proof. Define $L_0 = \bigcap\{L \in \mathscr{L} \colon W \cap L \neq 0\} \in \mathscr{L}$. It is sufficient to prove that $W \cap L_0 = \bigcap\{W \cap L \colon L \in \mathscr{L}$ and $W \cap L \neq 0\}$ is nonzero. Since $\mathscr{L}$ is nest and W finite-dimensional, for each L, $K \in \mathscr{L}$ we have $W \cap L = W \cap K$ if and only if $\dim W \cap L = \dim W \cap K$. So the set $\{W \cap L \colon L \in \mathscr{L}\}$ is finite and so the same holds for $\{W \cap L \colon L \in \mathscr{L}$ and $W \cap L \neq 0\}$. Since $\mathscr{L}$ is nest, $W \cap L_0 \neq 0$. □

Lemma 5. *Let X be a linear topological space and let $\mathscr{L}$ be a complete nest of subspaces of X. Let also W be a linear subspace of X with $\dim W = n$, $(n \in \mathbb{N})$. Then there exist $m \in \mathbb{N}$, and $\{n_j \colon j = 1, \dots, m\} \subseteq \mathbb{N}$ such that $n_1 + \cdots + n_m = n$, $\{w_j^i \colon i = 1, \dots, m,\ j = 1, \dots, n_i\} \subseteq W$ and $\{N_i \colon i = 1, \dots, m\} \subseteq L$ with the following properties: If we set $W_i = \langle w_j^k \colon k = i, i+1, \dots, m,\ j = 1, \dots, n_k\rangle$ $i = 1, \dots, m$ then*

(i) $W = \langle w_j^i \colon i = 1, \dots, m,\ j = 1, \dots, n_i\rangle$;

(ii) $W_i \cap N_i = \langle w_j^i \colon j = 1, \dots, n_i\rangle$ $i = 1, \dots, m$;

(iii) *For each* $i \in \{1, \ldots, m\}$ *and for each* $M \in \mathscr{L}$, *if* $W_i \cap M \neq 0$ *then* $N_i \subseteq x$; *and*

(iv) $0 \subset N_1 \subset N_2 \subset \cdots \subset N_m \subseteq X$.

Proof. Define $W_1 = W$. By Lemma 1 there is $N_1 \in \mathscr{L}$ such that $W_1 \cap N_1 \neq 0$ and if $W_1 \cap M \neq 0$ for $M \in \mathscr{L}$ then $N_1 \subseteq M$. Let $\{w_i^1 : i = 1, \ldots, n_1\}$ be a basis for $W_1 \cap N_1$. If $W_1 \cap N_1 = W_1$ simply take $m = 1$ and $n_1 = n$. If instead $\dim W_1 \cap N_1 = n_1 < n$, extend the preceding basis to a basis of W_1. Let then $\{w_i^1 : i = 1, \ldots, n\}$ be a basis of W_1. Define $W_2 = \langle w_i^1 : i = n_1 + 1, \ldots, n \rangle$ so that by Lemma 1 there is an $N_2 \in \mathscr{L}$ such that $W_2 \cap N_2 \neq 0$, and if $W_2 \cap M \neq 0$ for $M \in \mathscr{L}$ then $N_2 \subseteq M$. Let $\{w_i^2 : i = 1, \ldots, n_2\}$ be a basis for $W_2 \cap N_2$. If $W_2 \cap N_2 = W_2$ take $m = 2$ and $n_2 = n - n_1$. If $\dim(W_2 \cap N_2) = n_2 < n - n_1$ extend the preceding basis to a basis of W_2; that is let, $\{w_i^2 : i = 1, \ldots, n - n_1\}$ be a basis of W_2. Define $W_3 = \langle w_i^2 : i = n_2 + 1, \ldots, n - n_1 \rangle$ and continue in a similar manner. Since W is finite-dimensional, this process will terminate after a finite number, say m, steps. It is clear that it only remains to prove (iv). If on the contrary for some $i \in \{1, \ldots, m - 1\}$ we have $N_i \supseteq N_{i+1}$ we would then also have $W_i \cap N_i \supseteq W_i \cap N_{i+1} \supseteq W_{i+1} \cap N_{i+1} \neq 0$, a contradiction. This establishes the claim. □

Theorem 2. *Let* X *be a linear topological space,* $\mathscr{L}$ *a complete nest of subspaces of* X, *and* $R \in \mathrm{Alg}\mathscr{L}$ *be a finite rank operator of rank* n. *Then* R *can be written as a finite sum of* n *rank one operators each belonging to* $\mathrm{Alg}\mathscr{L}$.

Proof. By Lemma 2 applied to $W = R(X)$ there are $m \in \mathbb{N}$, $\{n_i : i = 1, \ldots, m\} \subseteq \mathbb{N}$, $\{w_j^i : i = 1, \ldots, m,\ j = 1, \ldots, n_i\} \subseteq R(X)$ and $\{N_i : i = 1, \ldots, m\} \subseteq \mathscr{L}$ satisfying the conclusion of that lemma. Therefore there is a set $\{x_{i,j}^* : i = 1, \ldots, m,\ j = 1, \ldots, n_j\} \subseteq X^*$ such that

$$R = \sum_{i=1}^{m} \sum_{j=1}^{n_i} x_{i,j}^* \otimes w_j^i .$$

We shall show that for $i_0 \in \{1, \ldots, m\}$ and $j_0 \in \{1, \ldots, n_{i_0}\}$ we have $x_{i_0, j_0}^* \otimes w_{j_0}^{i_0} \in \mathrm{Alg}\mathscr{L}$. It is sufficient (by Lemma 1) to prove that $x_{i_0, j_0}^*(N_{i_0} -) = 0$. Since $N_{i_0 -} = \bigvee\{M \in \mathscr{L} : M \subset N_{i_0}\} = \bigvee\{M \in \mathscr{L} : N_{i_0 - 1} \subseteq M \subset N_{i_0}\}$ it is sufficient to prove that if $M \in \mathscr{L}$ satisfies $N_{i_0 - 1} \subseteq M \subset N_{i_0}$ and if $x \in M$ then $x_{i_0, j_0}^*(x) = 0$. Define

$$z = R(x) - \left(\sum_{i=1}^{i_0 - 1} \sum_{j=1}^{n_i} x_{i,j}^*(x) w_j^i \right) = \sum_{i=i_0}^{m} \sum_{j=1}^{n_i} x_{i,j}^*(x) w_j^i .$$

From the first expression for z it follows that $z \in M \subset N_{i_0}$ (since $R \in \mathrm{Alg}\mathscr{L}$) and from second that $z \in W_{i_0}$. Therefore $z = 0$ and so $x_{i_0, j_0}^*(x) = 0$ and we are finished. □

The conclusion of Theorem 3 below, for the Hilbert space case, is in [4]. For more general completely distributive lattices it is shown in [8] that we have pointwise approximation of any operator in $\mathrm{Alg}\mathscr{L}$ by a sum of rank one operators, and the converse is also true. However note that this is weaker than density in the strong operator topology, and it has been an open question for some time if this is the case. A recent construction by Larson-Wogen (see "Note added in proof") in [12] shows that strong density may fail. For positive results in other special cases of completely distributive lattices see [2, 6, 13].

Theorem 3. *Let* $(X, \|\cdot\|)$ *be a normed space and let* $\mathcal{L}$ *be complete nest of subspaces of* X. *Then the closure of the set* $\mathcal{R} = \{R\colon R \in \mathrm{Alg}\mathcal{L}$ *and rank* $R < +\infty\}$ *in the strong operator topology is* $\mathrm{Alg}\mathcal{L}$.

Proof. Since $\mathcal{R}$ is an ideal of $\mathrm{Alg}\mathcal{L}$ it is sufficient to prove that we can approximate the identity. That is for each $\varepsilon > 0$, $n \in \mathbb{N}$ and for each $\{x_1, \dots, x_n\} \subseteq X$ linearly independent, there is an R_0 in $\mathrm{Alg}\mathcal{L}$ such that $\|x_i - R_0 x_i\| < \varepsilon$ $i = 1, \dots, n$. By Lemma 2 applied to $W = \langle x_1, \dots, x_n\rangle$ there are $m \in \mathbb{N}$, $\{n_i\colon i = 1, \dots, m\} \subseteq \mathbb{N}$, $\{w^i_j\colon i = 1, \dots, m,\ j = 1, \dots, n_i\} \subseteq \langle x_1, \dots, x_n\rangle$ and $\{N_i\colon i = 1, \dots, m\} \subseteq \mathcal{L}$ that satisfy the conclusion of the lemma.

It is sufficient to prove that for each $\varepsilon > 0$ there is an operator of finite rank, denoted by R, such that $\|w^i_j - Rw^i_j\| < \varepsilon$ $i = 1, \dots, m$, $j = 1, \dots, n_i$.

We shall find an R of the form

$$R = \sum_{i=1}^{m}\sum_{j=1}^{n_i} x^*_{i,j} \otimes y_{i,j}$$

where for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n_i\}$ we have $x^*_{i,j} \otimes y_{i,j} \in \mathrm{Alg}\mathcal{L}$, $\|w^i_j - (x^*_{i,j} \otimes y_{i,j})w^i_j\| < \varepsilon$ and $(x^*_{i,j} \otimes y_{i,j})w^k_t = 0$ if $k \neq i$ or $t \neq j$. Let $i_0 \in \{1, \dots, m\}$, $j_0 \in \{1, \dots, n_{i_0}\}$. Clearly there exist $L_k \in \mathcal{L}$ $k = 1, 2, \dots$ with $L_{k-} \subset N_{i_0}$ and there are $z_k \in L_k$ with $\lim z_k = w^{i_0}_{j_0}$. (This is so because if $N_{i_0 -} \subset N_{i_0}$ we could take $L_k = N_{i_0}$ and $z_k = w^{i_0}_{j_0}$. Otherwise $N_{i_0} = N_{i_0}- = \bigvee\{L\colon L \in \mathcal{L}$ and $L \subset N_{i_0}\}$ and the above follows). Each element of $L_{k-} \vee \langle w^i_j\colon i = i_0, i_0 + 1, \dots, m,\ j = 1, \dots, n_i\rangle$ can be written uniquely in the form

$$y + \sum_{i=i_0}^{m}\sum_{j=1}^{n_i} \lambda_{i,j} w^i_j \quad (\text{with } y \in L_{k-} \text{ and } \lambda_{i,j} \in \mathbb{C}) .$$

Indeed if $y + \sum_{i=i_0}^m \sum_{j=1}^{n_i} \lambda_{i,j} w^i_j = 0$ (with $y \in L_{k-}$) then

$$\sum_{i=i_0}^{m}\sum_{j=1}^{n_i} \lambda_{i,j} w^i_j = -y \in L_{k-} \subset N_{i_0} .$$

But $\sum_{i=i_0}^m \sum_{j=1}^{n_i} \lambda_{i,j} w^i_j \in W_{i_0}$, therefore $\sum_{i=i_0}^m \sum_{j=1}^{n_i} \lambda_{i,j} w^i_j = 0$, which proves the assertion.

Define now x^*_k $k = 1, 2, \dots$ by

$$x^*_k\left(y + \sum_{i=i_0}^{m}\sum_{j=1}^{n_i} \lambda_{i,j} w^i_j\right) = \lambda_{i_0, j_0} \quad (\text{for } y \in L_{k-})$$

and then extend it by the Hahn-Banach theorem to whole of X (so that $x^*_k \in X^*$). By Lemma 1, $x^*_k \otimes z_k$ belongs to $\mathrm{Alg}\mathcal{L}$. Also $(x^*_k \otimes z_k)w^{i_0}_{j_0} = z_k$ and $(x^*_k \otimes z_k)w^i_j = 0$ $i = i_0, \dots, m$, $j = 1, \dots, n_i$, $i \neq i_0$ or $j \neq j_0$. Moreover we have $N_{i_0 - 1} \subseteq L_{k-} \subset N_{i_0}$ for all large enough k because otherwise there would be a subsequence $\{L_{k_n}\}_{n=1}^{\infty}$ such that $L_{k_n} \subset N_{i_0 - 1}$ and, since $\lim z_k = w^{i_0}_{j_0}$, we would have $w^{i_0}_{j_0} \in N_{i_0 - 1}$, a contradiction. So, for all large enough k,

$(x_k^* \otimes z_k)w_j = 0$ $(i = 1, \dots, i_0 - 1,\ j = 1, \dots, n_i)$. We choose now k_0 so large as to satisfy the preceding condition and additionally $\|w_{j_0}^{i_0} - z_{k_0}\| < \varepsilon$. Define finally $x_{i_0, j_0}^* = x_{k_0}^*$, $y_{i_0, j_0} = z_{k_0}$ and we have finished. □

Remark. The above proof shows that if we are given n vectors then the approximating finite rank operator can be chosen to have rank at most n. This is an improvement to the rank $\frac{1}{2}n(n+1)$ of the original (Hilbert space) version of the theorem. It is clear that the n cannot be improved upon since it is well known that for sufficiently small ε, the linear independence of vectors $\{x_i\}_{i=1}^n$ implies the linear independence of any set $\{y_i\}_{i=1}^n$ of vectors with $\|x_i - y_i\| < \varepsilon$ $(1 \le i \le n)$.

Also note that if the underlying space is a Hilbert space, then Erdos [4] shows that the identity can be approximated in the strong operator topology by a finite rank operator of $\operatorname{Alg}\mathscr{L}$ taken from the *unit ball* of $B(X)$. A similar unit ball conclusion (for separable Hilbert space) is valid for a Boolean lattice of two subspaces, as shown in [9]. Both these theorems make elaborate use of Hilbert space techniques. In general normed spaces (even in Banach spaces) this fails. For example if the Banach space fails the bounded approximation property, it is not possible to approximate the identity in the strong operator topology. It is an open problem if this unit ball density is valid when the underlying space has the metric approximation property.

Added in proof. It has been brought to the author's attention that Theorem 3 above and the subsequent remark answer a recent question of Han Deguang, *Rank One Operators and Bimodules of Reflexive Operator Algebras in Banach spaces*, J. Math. Anal. and Appl., **161** (1991) 188-193, who proves our theorem for the $n = 2$ special case.

Lately there is an interest in versions of Lances' theorem, stated above, viewing it as an interpolation problem. For example, generalizations (different to our Theorem 1) appear in M. Anoussis, *Interpolating Operators in Next Algebras* (preprint), in E. G. Katsoulis, R. L. Moore, T. T. Trent, *Interpolation in Nest algebras and applications to operator corona theorems* (to appear) J. Oper. Th., and in the references given there.

Acknowledgment

The author wishes to thank M. S. Lambrou without whose help it would have been impossible to write this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRETE, 714 09 IRAKLIO, CRETE, GREECE