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ON THE INDECOMPOSABILITY OF COMPACT CONVEX SETS

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ABSTRACT. Let C be a compact convex set in a Hausdorff, locally convex linear topological space X and let $\mathscr{F}$ be the family of affine homeomorphisms of X onto itself. It is proved that C is indecomposable under $\mathscr{F}$; i.e. if $C = A \cup B$ and $B = F[A]$, for some $F \in \mathscr{F}$, then $A \cap B \neq \varnothing$.

INTRODUCTION

A set S in a metric space X is said to be decomposable, under the family of isometries of X onto itself, if it is the disjoint union of two isometric subsets. In [1] it was shown that weakly compact convex subsets of a Banach space are not decomposable under the affine isometries of the space onto itself.

It is the purpose of this paper to prove the stronger

Theorem. *Let C be a compact convex subset of a Hausdorff, locally convex linear topological space X. Then C is not decomposable under the family of affine homeomorphisms of X onto itself.*

(For a brief history of closely related results see [1]; for general background material see the comprehensive survey [2] by Stan Wagon.)

Proof of the theorem. Let $\mathscr{F}$ be the family of affine homeomorphisms of X onto itself. Let $F \in \mathscr{F}$ and suppose that $C = A \cup B$ and $B = F[A]$. We have to prove that $A \cap B \neq \varnothing$. Let $0(x) = \{F^n(x): \ n \geq 0\}$. Two mutually exclusive cases arise:

(i) there is an $x \in C$ such that $0(x) \subset C$; and
(ii) there is no such x.

In case (i), since F is both affine and continuous, $\overline{co}(0(x))$ is mapped into itself, and by known theorems, e.g., one by Tychonov [1, p. 107], a $\xi \in C$ exists such that $F(\xi) = \xi$. It then readily follows that ξ is common to A and B. In case (ii), to any $x \in C$ there is a positive integer n such that $F^n(x) \in X \backslash C$. Let $\rho: C \to \mathbb{N}$ be the mapping sending $x \in C$ to the least $n \in \mathbb{N}$ with the property that $F^n(x) \in X \backslash C$. As can be readily seen, ρ is bounded. (If not, we may for each $n \in \mathbb{N}$ choose an x with $\rho(x) \geq n$. The set D so obtained has at least one accumulation point $\overline{x}$. If $\rho(\overline{x}) = N$ then

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$F^N(\overline{x}) \in X\backslash C$. However, $F^{-N}[X\backslash C]$ is a neighbourhood of $\overline{x}$ that contains some y with $\rho(y) \geq N+1$; so $\rho(y) \notin X\backslash C$, a contradiction.) Let $N = \sup \rho[C]$ and let $M = \{x \in C : \rho(x) = N\}$. Clearly $M \neq \varnothing$ and if $x \in M$, then $F^{-1}(x) \in X\backslash C$. (Otherwise, with $u = F^{-1}(x)$, $u, F(u), \ldots, F^n(u)$ are all members of C, and $\rho(u) \geq N+1$.) This fact, in turn, implies that $x \in A$ as B is the one-one image of A.

Let $P(x)$ be the polygonal arc joining consecutive members of $F^{-1}(x), x, \ldots, F^N(x)$ with the closed line segments $[F^{k-1}(x), F^k(x)]$ for $0 \leq k \leq N$. If $x \in M$, $P(x)$ has the special properties:

(1) $[F^{-1}(x), x] \cap B = \varnothing$ and
(2) $[F^{N-1}(x), F^N(x)] \cap A = \varnothing$.

Indeed, if $u = \lambda x + (1-\lambda)F^{-1}(x) \in B$ for some λ, $0 < \lambda < 1$, then $F^{-1}(u) \in A \subset C$ and, more generally, $F^k(u) = \lambda F^k(x) + (1-\lambda)F^{k-1}(x) \in C$ for $0 \leq k \leq N-1$. Hence $u \in B$ implies that $\{F^{-1}(u), u, \ldots, F^{N-1}(u)\} \subset C$ and, therefore, $\rho(F^{-1}(u)) \geq N+1$ against the maximality property of M. An analogous argument proves (2).

It follows that $[F^{-1}(x), x] \cap C = [y, x]$, where $[y, x]$ is a closed line segment in A with end points y, x. Similarly, $[F^{N-1}(x), F^N(x)] \cap C = [F^{N-1}(x), z]$ is a closed line segment in B with end points $F^{N-1}(x), z$. The polygonal arc $P^* \subset P(x)$ with end points y, z is, thus, a subset of C and $A \cap P^*$ is mapped by F onto $B \cap P^*$. We claim that P^* is a simple arc; i.e., two line segments

$$[F^m(x), F^{m+1}(x)] \cap C \quad \text{and} \quad [F^n(x), F^{n+1}(x)] \cap C,$$

with $n \geq m+2$, are disjoint. To prove this claim observe, first, that if $u \in [x, F(x)]$ then $\rho(u) \geq N-1$ since $[F^{N-1}(x), F^{N-1}(x)] \subset C$. Suppose now that $[F^m(x), F^{m+1}(x)] \cap C$ intersects $[F^n(x), F^{n+1}(x)]$. (Clearly (1) and (2) imply that either $m \neq -1$ or $n \neq N-1$; and it readily follows that $n - m \leq N-1$.) Applying F^{-n} to both segments there must be a point v that is common to $[F^{m-n}(x), F^{m+1-n}(x)]$ and $[x, F(x)]$. Further, $v = F^{m-n}(u)$ for some $u \in [x, F(x)]$ and therefore

$$\rho(v) = \rho(u) + n - m \geq N - 1 + 2 = N + 1,$$

contradicting the maximality property of N.

We conclude that P^* is a homeomorph of $[0, 1] \subset \mathbb{R}$ in the usual topology. Hence the natural order of $[0, 1]$ induces a corresponding order on P^*, with 0 corresponding to y and 1 to z. Thus it makes sense to refer, for example, to a left-closed, right-open $[\cdot, \cdot)$ subarc. Let K_0 be the subarc of P^* extending from y to $F(y)$ (with $F(y)$ removed, i.e., $K_0 = [y, x] \cup [x, F(y))$). As noted above $[y, x] \subset A$; and similarly, $[x, F(y))$, as the image of $[F^{-1}(x), y) \subset X\backslash C$, is a subset of A. Hence $K_0 \subset A$ and x is in its relative interior. Thus $A \cap B = \varnothing$ implies that $K_0 \cap B = \varnothing$. Let $K_1 = F[K_0]$. Then $K_1 \cap K_0 = \varnothing$ and K_1 is of the form $[F(y), F(x)] \cup [F(x), F^2(y))$, i.e., the left-closed, right-open subarc $[F(y), F^2(y))$; and $F(x)$ is in its relative interior. This type of configuration is preserved under F and terminates in $K_{N-1} \subset B$. Thus $P^* \cap A = K_0 \cup K_2 \cup \cdots \cup K_{N-2}$, while its image under F is $K_1 \cup K_3 \cup \cdots \cup K_{N-1}$. Hence the assumption that $A \cap B = \varnothing$ leads to the absurd conclusion that P^* is the finite disjoint union of left-closed, right-open subarcs K_i, $0 \leq i \leq N-1$. Hence $A \cap B \neq \varnothing$, completing the proof of the theorem. □

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