A NOTE ON VECTOR-VALUED HARDY AND PALEY INEQUALITIES

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Abstract. The values of $p$ and $q$ for $L_p(L_q)$ that satisfy the extension of Paley and Hardy inequalities for vector-valued $H^1$ functions are characterized. In particular, it is shown that $L_2(L_1)$ is a Paley space that fails Hardy inequality.

Introduction

In [BP] the vector-valued analogue of two classical inequalities in the theory of Hardy spaces were investigated. A complex Banach space $X$ is said to be a Paley space if

\[(P) \quad \left( \sum_{k=0}^{\infty} \| \hat{f}(2^k) \|^2 \right)^{1/2} \leq C \| f \|_1 \quad \text{for all } f \in H^1(X).\]

A complex Banach space $X$ is said to verify vector-valued Hardy inequality (for short $X$ is a (H1)-space) if

\[(H) \quad \sum_{n=0}^{\infty} \frac{\| \hat{f}(n) \|}{n+1} \leq C \| f \|_1 \quad \text{for all } f \in H^1(X),\]

where $H^1(X) = \{ f \in L^1(T, X) : \hat{f}(n) = 0 \text{ for } n < 0 \}$.

Both inequalities can be regarded in the framework of vector-valued extensions of multipliers from $H^1$ to $l^1$. Recall that a sequence $(m_n)$ is a $(H^1 - l^1)$-multiplier, to be denoted by $m_n \in (H^1 - l^1)$, if $T_{m_n}(f) = (\hat{f}(n) m_n)$ defines a bounded operator from $H^1$ into $l^1$.

The $(H^1 - l^1)$-multipliers were characterized by C. Fefferman in the following way (see [SW] for a proof):

\[(*) \quad (H^1 - l^1) = \left\{ m_n : \sup_{s \geq 1} \left( \sum_{k \geq 1} \left( \sum_{j=ks+1}^{(k+1)s} |m_j| \right)^2 \right)^{1/2} < \infty \right\}.

\]

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A complex Banach space is said to have \((H^1 - l^1)\)-Fourier type if
\[
(F) \sum_{n=0}^{\infty} \|\hat{f}(n)\| m_n \leq C\|f\|_1 \quad \text{for all } f \in H^1(X) \text{ and for all } m_n \in (H^1 - l^1).
\]

The reader is referred to [BP] for examples of spaces having and failing these properties and for their connection with the notions of Rademacher type and Fourier type.

Using \((*)\) it is easy to see that any space of \((H^1 - l^1)\)-Fourier type must be a Paley and a \((H1)\)-space. Unfortunately the only examples of spaces without \((H^1 - l^1)\)-Fourier type that we had at our disposal also behave badly with respect to the other two properties. The problem of finding a Paley space failing Hardy inequality or without \((H^1 - l^1)\)-Fourier type was left open (see [BP, Remark 4.1]).

Surprisingly it is enough to deal with Lebesgue spaces of mixed norm, namely, \(L_p(L_q)\), to produce a simple example of Paley space failing Hardy inequality. In fact we shall see that \(L_2(L_1)\) is such an example.

Given \(1 \leq p \leq \infty\), \((\Omega, \Sigma, \mu)\) a \(\sigma\)-finite measure space, and a Banach space \(Y\) we denote by \(L_p(\mu, Y)\) the space of \(Y\)-valued strongly measurable functions such that \(\|f\| \in L_p(\mu)\).

Throughout the paper \(1 \leq p, q \leq \infty\) and we shall use the notation \(L_p(L_q) = L_p(T, L_q(T))\).

**Paley spaces**

For self-containedness of the paper, we provide here simple direct proofs of special cases of Corollary 3.2 and Theorem 3.2 of [BP] that show how the Paley property behaves with respect to the vector-valued extension.

**Lemma 1.** Let \(1 \leq p < 2\), \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and \(Y\) a Paley space. Then \(L_p(\mu, Y)\) is a Paley space.

**Proof.** The case \(p = 2\) is a simple consequence of Fubini's theorem. Let us assume \(1 \leq p < 2\) and \(q = (2/p)' = 2/(2 - p)\). Let us take \(f(t) = \sum_{n \geq 0} x_n e^{it}\) where \(x_n \in L_p(\mu, Y)\).

\[
\left( \sum_{k \geq 0} \|x_{2^k}\|^2_{L_p(\mu, Y)} \right)^{1/2} = \left( \sum_{k \geq 0} \left( \int_{\Omega} \|x_{2^k}(w)\|^p_Y d\mu(w) \right)^{2/p} \right)^{1/2}
\]

\[
\leq \sup_{\alpha_1^2 = 1} \left( \sum_{k \geq 0} \int_{\Omega} \|x_{2^k}(w)\|^p_Y \alpha_k d\mu(w) \right)^{1/p}
\]

\[
\leq \left( \int_{\Omega} \left( \sum_{k \geq 0} \|x_{2^k}(w)\|^2_Y \right)^{p/2} d\mu(w) \right)^{1/p}
\]

\[
\leq C \left( \int_{\Omega} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \| \sum_{n \geq 0} x_n(w) e^{int} \|_Y dt \right)^p d\mu(w) \right)^{1/p}
\]
Theorem 1. \( L_p(L_q) \) is a Paley space if and only if \( 1 \leq p, q \leq 2 \).

Proof. It is clear from the definition that a Paley space must have cotype 2. (Recall that the notion of cotype can be defined with \( e^{it^2} \) instead of Rademacher functions.) Now the cotype 2 condition forces the values of \( 1 \leq p, q \leq 2 \).

To get the converse, observe that the classical Paley inequality together with Lemma 1 for \( Y = \mathbb{C} \) gives that \( L_q \) is a Paley space for \( 1 \leq q \leq 2 \). Now apply Lemma 1 again. \( \square \)

(HI)-SPACES

Let us now prove the main result of the paper.

Theorem 2. If \( 1 < p < \infty \) then \( l_p(H^1) \) is not a (HI)-space.

Proof. The case \( p = \infty \) is immediate since it does contain \( c_0 \) and \( c_0 \) fails Hardy (see [BP]). We assume then \( 1 < p < \infty \). Let us consider the function

\[
f(z) = \frac{1}{(1 - z)^p \log(1 - z)^{-1}} = \sum_{n=0} a_n z^n.
\]

Let us define \( \Phi(z) = \lim_{r \to 1} \Phi(rz) \) for \( |z| = 1 \) if the radial limit exists. We shall show that \( \Phi \in H^1(l_p(H^1)) \).

\[
\Phi(z)(k, w) = 2^{k(1-p)} \sum_{n=0} a_n (1 - \frac{1}{2^k})^n w^n z^n = 2^{k(1-p)} f((1 - \frac{1}{2^k}) w z).
\]

Using (2) we have

\[
\|\Phi(z)\|_{l_p(H^1)} = \left( \sum_{k=1}^{\infty} 2^{k(1-p)} M_p^p(f, (1 - 2^{-k})|z|) \right)^{1/p} \leq C \left( \sum_{k=1}^{\infty} \frac{1}{k^p} \right)^{1/p} < \infty.
\]
Now since $\Phi$ is uniformly bounded then $\sup_{0<r<1} M_1(\|\Phi\|, r) < \infty$. Using the fact that for $1 \leq p < \infty$ the Banach space $l_p(H^1)$ is a separable dual by a routine argument, we show that the radial limit $F(z)$ exists almost everywhere and that $F \in H^1(l_p(H^1))$ for $1 < p < \infty$.

On the other hand

$$\|x_n\|_{l_p(H^1)} = a_n \left( \sum_{k=1}^{\infty} 2^{kp(1-p)} \left(1 - \frac{1}{2^k}\right)^{np} \right)^{1/p} \geq a_n \left( \sum_{k \geq \log_2 n} 2^{kp(1-p)} \left(1 - \frac{1}{n}\right)^{np} \right)^{1/p}$$

Since $(1 - 1/n)^{np}$ converges to $e^{-p}$, for $n$ big enough we have

$$\|x_n\|_{l_p(H^1)} \geq C_p a_n \left( \sum_{k \geq \log_2 n} 2^{kp(1-p)} \right)^{1/p} \geq C_p a_n n^{1-p}.$$

Now using estimate (1) we get $\sum_{n=1}^{\infty} \|x_n\|_{l_p(H^1)}/(n+1) = \infty$. □

**Remark.** If $1 < p < \infty$ then $l_p(H^1)$ is a Paley space but is not a (HI)-space. (Hence it does not have $(H^1 - l^q)$-Fourier type.)

**Theorem 3.** $L_p(L_q)$ is a (HI)-space if and only if either $1 < p, q < \infty$ or $p = 1$ and $1 \leq q < \infty$.

**Proof.** Let us first show that under such assumptions on $p, q$ we get (HI)-spaces. It is an application of Fubini's theorem that if $Y$ is a (HI)-space then $L_1(\mu, Y)$ is a (HI)-space. Combining this with the result that every $B$-convex space (Rademacher type bigger than 1) is a (HI)-space (see [BP, Bo]) we get this implication.

For the other implication observe that the cases $p = \infty$ or $q = \infty$ must be excluded because then $L_p(L_q)$ would contain $c_0$. The case $q = 1$ follows from Theorem 1, since $l_p$ embeds into $L_p(\mathbb{T})$ and $H^1$ into $L_1(\mathbb{T})$. □

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**References**


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