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A NOTE ON VECTOR-VALUED HARDY AND PALEY INEQUALITIES

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ABSTRACT. The values of p and q for $L_p(L_q)$ that satisfy the extension of Paley and Hardy inequalities for vector-valued H^1 functions are characterized. In particular, it is shown that $L_2(L_1)$ is a Paley space that fails Hardy inequality.

INTRODUCTION

In [BP] the vector-valued analogue of two classical inequalities in the theory of Hardy spaces were investigated. A complex Banach space X is said to be a Paley space if

$$\text{(P)} \qquad \left(\sum_{k=0}^{\infty} \|\hat{f}(2^k)\|^2\right)^{1/2} \le C\|f\|_1 \quad \text{for all } f \in H^1(X).$$

A complex Banach space X is said to verify vector-valued Hardy inequality (for short X is a (HI)-space) if

$$\text{(H)} \qquad \sum_{n=0}^{\infty} \frac{\|\hat{f}(n)\|}{n+1} \le C\|f\|_1 \quad \text{for all } f \in H^1(X),$$

where $H^1(X) = \{f \in L^1(\mathbb{T}, X) \colon \hat{f}(n) = 0 \text{ } for \text{ } n < 0\}$.

Both inequalities can be regarded in the framework of vector-valued extensions of multipliers from H^1 to l^1. Recall that a sequence (m_n) is a $(H^1 - l^1)$-multiplier, to be denoted by $m_n \in (H^1 - l^1)$, if $T_{m_n}(f) = (\hat{f}(n)m_n)$ defines a bounded operator from H^1 into l^1.

The $(H^1 - l^1)$-multipliers were characterized by C. Fefferman in the following way (see [SW] for a proof):

$$(*) \qquad (H^1 - l^1) = \left\{m_n \colon \sup_{s\ge 1} \left(\sum_{k\ge 1} \left(\sum_{j=ks+1}^{(k+1)s} |m_j|\right)^2\right)^{1/2} < \infty\right\}.$$

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A complex Banach space is said to have $(H^1 - l^1)$-Fourier type if

$$\text{(F)}\quad \sum_{n=0}^{\infty} \|\hat{f}(n)\| |m_n| \leq C\|f\|_1 \quad \text{for all } f \in H^1(X) \text{ and for all } m_n \in (H^1 - l^1).$$

The reader is referred to [BP] for examples of spaces having and failing these properties and for their connection with the notions of Rademacher type and Fourier type.

Using $(*)$ it is easy to see that any space of $(H^1 - l^1)$-Fourier type must be a Paley and a (HI)-space. Unfortunately the only examples of spaces without $(H^1 - l^1)$-Fourier type that we had at our disposal also behave badly with respect to the other two properties. The problem of finding a Paley space failing Hardy inequality or without $(H^1 - l^1)$-Fourier type was left open (see [BP, Remark 4.1]).

Surprisingly it is enough to deal with Lebesgue spaces of mixed norm, namely, $L_p(L_q)$, to produce a simple example of Paley space failing Hardy inequality. In fact we shall see that $L_2(L_1)$ is such an example.

Given $1 \leq p \leq \infty$, (Ω, Σ, μ) a σ-finite measure space, and a Banach space Y we denote by $L_p(\mu, Y)$ the space of Y-valued strongly measurable functions such that $\|f\| \in L_p(\mu)$.

Throughout the paper $1 \leq p, q \leq \infty$ and we shall use the notation $L_p(L_q) = L_p(\mathbb{T}, L_q(\mathbb{T}))$.

Paley spaces

For self-containedness of the paper, we provide here simple direct proofs of special cases of Corollary 3.2 and Theorem 3.2 of [BP] that show how the Paley property behaves with respect to the vector-valued extension.

Lemma 1. *Let $1 \leq p \leq 2$, (Ω, Σ, μ) be a σ-finite measure space and Y a Paley space. Then $L_p(\mu, Y)$ is a Paley space.*

Proof. The case $p = 2$ is a simple consequence of Fubini's theorem. Let us assume $1 \leq p < 2$ and $q = (2/p)' = 2/(2-p)$. Let us take $f(t) = \sum_{n\geq 0} x_n e \int$ where $x_n \in L_p(\mu, Y)$.

$$\left(\sum_{k\geq 0} \|x_{2^k}\|^2_{L_p(\mu, Y)}\right)^{1/2} = \left(\sum_{k\geq 0} \left(\int_\Omega \|x_{2^k}(w)\|^p_Y d\mu(w)\right)^{2/p}\right)^{1/2}$$

$$= \sup_{\sum \alpha_k^q = 1} \left(\sum_{k\geq 0} \int_\Omega \|x_{2^k}(w)\|^p_Y \alpha_k d\mu(w)\right)^{1/p}$$

$$\leq \left(\int_\Omega \left(\sum_{k\geq 0} \|x_{2^k}(w)\|^2_Y\right)^{p/2} d\mu(w)\right)^{1/p}$$

$$\leq C\left(\int_\Omega \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\sum_{n\geq 0} x_n(w) e^{\text{int}}|_Y dt\right)^p d\mu(w)\right)^{1/p}$$

$$= C \sup_{\|h\|_{p'}=1} \int_\Omega \int_{-\pi}^{\pi} \|\sum_{n\geq 0} x_n(w)e^{\mathrm{int}}\|_Y h(w)\, dt\, d\mu(w)$$

$$\leq C\int_{-\pi}^{\pi} \|\sum_{n\geq 0} x_n(w)e^{\mathrm{int}}\|_{L_p(\mu, Y)}\, dt = C\int_{-\pi}^{\pi} \|f(t)\|_{L_{p(\mu,Y)}}\, dt. \quad \square$$

Theorem 1. *$L_p(L_q)$ is a Paley space if and only if $1 \leq p, q \leq 2$.*

Proof. It is clear from the definition that a Paley space must have cotype 2. (Recall that the notion of cotype can be defined with $e^{i2^k t}$ instead of Rademacher functions.) Now the cotype 2 condition forces the values of $1 \leq p, q \leq 2$.

To get the converse, observe that the classical Paley inequality together with Lemma 1 for $Y = \mathbb{C}$ gives that L_q is a Paley space for $1 \leq q \leq 2$. Now apply Lemma 1 again. $\square$

(HI)-SPACES

Let us now prove the main result of the paper.

Theorem 2. *If $1 < p \leq \infty$ then $l_p(H^1)$ is not a (HI)-space.*

Proof. The case $p = \infty$ is immediate since it does contain c_0 and c_0 fails Hardy (see [BP]). We assume then $1 < p < \infty$. Let us consider the function

$$f(z) = \frac{1}{(1-z)^p} \frac{z}{\log(1-z)^{-1}} = \sum_{n=0} a_n z^n.$$

Let us recall the following estimates (see [L p. 93–96]

$$a_n \sim \frac{n^{p-1}}{\log n} \qquad (n \to \infty), \tag{1}$$

$$M_1(f, r) \sim \frac{(1-r)^{1-p}}{\log(1-r)^{-1}} \qquad (r \to 1), \tag{2}$$

where $M_1(f, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|dt$.

Let us define $\Phi(z) = \sum_{n=0}^\infty x_n z^n$ for $|z| < 1$ where $x_n \in l_p(H^1)$ are given by

$$x_n(k, w) = \left(2^{k(1-p)} a_n (1 - 1/2^k)^n w^n\right) \qquad (k \in \mathbb{N},\ |w| = 1).$$

Note that $\|x_n\|_{l_p(H^1)} \leq Ca_n$ shows that Φ is analytic on the open unit disc and takes values in $l_p(H^1)$.

Let us define $F(z) = \lim_{r\to 1} \Phi(rz)$ for $|z| = 1$ if the radial limit exists. We shall show that $F \in H^1(l_p(H^1))$.

$$\Phi(z)(k, w) = 2^{k(1-p)} \sum_{n=0}^{\infty} a_n (1 - \frac{1}{2^k})^n w^n z^n = 2^{k(1-p)} f((1 - \frac{1}{2^k})wz).$$

Using (2) we have

$$\|\Phi(z)\|_{l_p(H^1)} = \left(\sum_{k=1}^\infty 2^{kp(1-p)} M_1^p(f, (1-2^{-k})|z|)\right)^{1/p} \leq C\left(\sum_{k=1}^\infty \frac{1}{k^p}\right)^{1/p} < \infty.$$

Now since Φ is uniformly bounded then $\sup_{0\le r<1} M_1(\|\Phi\|, r) < \infty$. Using the fact that for $1 \le p < \infty$ the Banach space $l_p(H^1)$ is a separable dual by a routine argument, we show that the radial limit $F(z)$ exists almost everywhere and that $F \in H^1(l_p(H^1))$ for $1 < p < \infty$.

On the other hand

$$\|x_n\|_{l_p(H^1)} = a_n \left(\sum_{k=1}^{\infty} 2^{kp(1-p)}(1 - \frac{1}{2^k})^{np} \right)^{1/p} \ge a_n \left(\sum_{k \ge \log_2 n}^{\infty} 2^{kp(1-p)}(1 - \frac{1}{n})^{np} \right)^{1/p}$$

Since $(1 - 1/n)^{np}$ converges to e^{-p}, for n big enough we have

$$\|x_n\|_{l_p(H^1)} \ge C_p a_n \left(\sum_{k \ge \log_2 n}^{\infty} 2^{kp(1-p)} \right)^{1/p} \ge C_p a_n n^{1-p}.$$

Now using estimate (1) we get $\sum_{n=1}^{\infty} \|x_n\|_{l_p(H^1)}/(n+1) = \infty$. □

Remark. If $1 < p < \infty$ then $l_p(H_1)$ is a Paley space but is not a (HI)-space. (Hence it does not have $(H^1 - l^q)$-Fourier type.)

Theorem 3. *$L_p(L_q)$ is a (HI)-space if and only if either $1 < p, q < \infty$ or $p = 1$ and $1 \le q < \infty$.*

Proof. Let us first show that under such assumptions on p, q we get (HI)-spaces. It is an application of Fubini's theorem that if Y is a (HI)-space then $L_1(\mu, Y)$ is a (HI)-space. Combining this with the result that every B-convex space (Rademacher type bigger than 1) is a (HI)-space (see [BP, Bo]) we get this implication.

For the other implication observe that the cases $p = \infty$ or $q = \infty$ must be excluded because then $L_p(L_q)$ would contain c_0. The case $q = 1$ follows from Theorem 1, since l_p embbeds into $L_p(\mathbb{T})$ and H^1 into $L_1(\mathbb{T})$. □

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