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A NOTE ON DETERMINACY

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ABSTRACT. In this paper, we present a particularly simple and direct proof that the set of noncontact-sufficient ($\mathscr{K}$-sufficient) germs are of infinite codimension. Our proof gives, for each k, an integer r with the property that almost all r-jets over any k-jet z is $\mathscr{K}$-sufficient. Similar results are obtained for $\mathscr{A}$ or right-left equivalence when the source and target dimensions (n, p) are $(2, 2)$ and $(2, 3)$.

In this paper we use a surprising fact (Lemma 1.2) to prove some results concerning $\mathscr{A}$- and $\mathscr{K}$-determinacy. (See [9] for the notions of $\mathscr{A}$ and $\mathscr{K}$ equivalence and determinacy.) We present a particularly simple and direct proof that the set of non-$\mathscr{K}$-sufficient germs are of infinite codimension, which gives for each k an integer r with the property that almost all r-jets over any k-jet z is $\mathscr{K}$-sufficient. The estimate is of the same order in k (k^n, where n is the dimension of the source of the jet) as that obtained by Wirthmüller in [10] and consequently is a vast improvement on the estimate in [1]. The coefficient of this leading term in [10] is considerably smaller than that given here, but our proof is much simpler and more direct.

Our technique comes into its own when we turn to $\mathscr{A}$-determinacy. The range of dimension (n, p) in which nonfinitely $\mathscr{A}$-determined germs $\mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ are of infinite codimension has been determined by duPlessis in [8]. Here we present a direct proof for certain pairs of dimensions (n, p); namely, $(n, 1)$ (rather trivial); (2, 2); (2, 3); (n, p), $p \geq 2n - 1$. The advantage of our approach is that in all bar the last case it also yields estimates for the least integer r with the property that almost all r-jets over any k-jet z is $\mathscr{A}$-sufficient. We suspect that in both cases ($\mathscr{K}$ and $\mathscr{A}$) the known estimates are far too large.

Notation. We use the terminology and notation of [8].

1. The $\mathscr{K}$ case

In what follows $\mathscr{G}$ is one of Mather's groups $\mathscr{A}$ or $\mathscr{K}$. We need the following

Fact 1.1. The property of being finitely $\mathscr{G}$-determined is open. More precisely,

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if $f(t, x)\colon \mathbb{R}^{n+1}, 0 \to \mathbf{R}^p, 0$ is smooth with $f(t, 0) = 0$ and $f(0, x) = f_0(x)\colon R^n, 0 \to \mathbf{R}^p, 0$ is finitely $\mathcal{G}$-determined, then $f_t\colon \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is finitely $\mathcal{G}$-determined for all t sufficiently small. (For a proof in the case of $\mathcal{G} = \mathcal{A}$ see [3, p. 306]. The case $\mathcal{G} = \mathcal{K}$ is easier.)

Now we state the crucial

Lemma 1.2. *Assign positive integer weights w_i to x_i, and suppose that $f_i(x_1, \dots, x_n)$ is homogeneous of total weight d_i, $1 \le i \le p$, and that $f = (f_1, \dots, f_p)\colon \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is finitely $\mathcal{G}$-determined. If $g = (g_1, \dots, g_p)\colon \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is a polynomial mapping with all terms of g_i of weight $< d_i$, then $f + g\colon \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is finitely $\mathcal{G}$-determined. Moreover, $d_e(f, \mathcal{G}) \ge d_e(f + g, \mathcal{G})$.*

Proof. Let t be a positive real number. Then defining $\phi_t\colon \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ by $\phi_t(x) = (t^{-w_1}x_1, \dots, t^{-w_n}x_n)$, and $\psi_t\colon \mathbf{R}^p, 0 \to \mathbf{R}^p, 0$ by $\psi_t(y) = (t^{d_1}y_1, \dots, t^{d_p}y_p)$, we have $\psi_t \circ f \circ \phi_t(x) = f(x)$. So $f_t = \psi_t \circ (f + g) \circ \phi_t = f + \psi_t \circ g \circ \phi_t$ and $f_0 = f$. So for t small f_t is finitely $\mathcal{G}$-determined (Fact 1.1). But f_t is $\mathcal{G}$-equivalent to $f + g$, so $f + g$ is finitely $\mathcal{G}$-determined. The last assertion follows because for t small, $d_e(f, \mathcal{G}) \ge d_e(f_t, \mathcal{G})$. □

Our first application is to $\mathcal{K}$-determinacy.

Proposition 1.3. (i) *Finite $\mathcal{K}$-determinacy holds in general.*

(ii) (a) *If $n \le p$ and $r \ge (k+1)^{n-1}(p(k+1) - n) + 1$, then most r-jets over a given k-jet are $\mathcal{K}$-sufficient.* (b) *If $n \ge p$ and $r \ge (-1)^{n-p+1}\{1 + (k+1)^p \sum_{j=0}^{n-p}(-1)^{j+1}\binom{p+j-1}{j}\binom{n}{n-p-j}(k+1)^j\} + 1$, then most r-jets over a given k-jet are $\mathcal{K}$-sufficient.*

Proof. (i) Since the non-$\mathcal{K}$-finite germs form a proalgebraic set, we have only to produce for each k-jet z a $\mathcal{K}$-finite germ f with $j^k f = z$. This is normally done by applying Sard's theorem, but clearly using Lemma 1.2, we can write f down quite explicitly. We need only find a $\mathcal{K}$-finite homogeneous jet of degree $k + 1$. If $n \le p$ the map $h_{k+1}(x_1, \dots, x_n) = (x_1^{k+1}, \dots, x_n^{k+1}, 0, \dots, 0)$ is $\mathcal{K}$-finite. If $n \ge p$ then we need to find a homogeneous mapping $\mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ defining a smooth complete intersection in $\mathbf{C}P^{n-1}$. Let $\lambda_1, \dots, \lambda_n$ be distinct nonzero real numbers, and define $h(x) = (\lambda_1^0 x_1^{k+1} + \cdots + \lambda_n^0 x_n^{k+1}, \lambda_1^1 x_n^{k+1} + \cdots + \lambda_n^1 x_n^{k+1}, \dots, \lambda_1^{p-1} x_1^{k+1} + \cdots + \lambda_n^{p-1} x_n^{k+1})$. We prove, by induction on $n - p$, that h is of F.S.T. If $n = p$ then h is equivalent by a linear change of coordinates in the target to $(x_1^{k+1}, \dots, x_n^{k+1})$, which is of F.S.T. Now suppose that we have established the cases $n - p \le r$ and that $n = p + r + 1$. If h is not of F.S.T. we can find some $x \ne 0$ with $h(x) = 0$ and $dh(x)$ having rank $< p$. Looking at the first $p \times p$ minor of $dh(x)$, we deduce that $x_1 x_2 \cdots x_p = 0$. Without loss of generality, suppose that $x_1 = 0$. Then setting $\overline{h} = h|\{x_1 = 0\}$, $\overline{x} = (x_2, \dots, x_n)$, we find that $\overline{h}(\overline{x}) = 0$ and $d\overline{h}(\overline{x})$ has rank $< p$, contradicting our inductive hypothesis.

(ii) In the case $n \le p$ it is clear that the $\mathcal{K}_e$-codimension of h_{k+1} is $p(k+1)^n - n(k+1)^{n-1}$, and so the $\mathcal{K}_e$-codimension of $F = z + h_{k+1}$ is $\le (k+1)^{n-1}(p(k+1) - n)$. By Nakayama's Lemma [9, p. 489], since $d_e(F, \mathcal{K}) = \dim_{\mathbf{R}} V(F)/T\mathcal{K}_e f \le (k+1)^{n-1}(p(k+1) - n)$, we have $\mathcal{M}_n^{(k+1)^{n-1}(p(k+1)-n)} \cdot V(F) \subset T\mathcal{K}_e F$, and F is $(k+1)^{n-1}(p(k+1) - n) + 1 - \mathcal{K}$-determined. So we can find an r-jet over z (namely, F) that is $\mathcal{K}$-sufficient if $r \ge$

$(k+1)^{n-1}(p(k+1)-n)+1$, and given such an r, any r-jet g with $d_e(g, \mathscr{K}) \leq (k+1)^{n-1}(p(k+1)-n)$ is $\mathscr{K}$-sufficient. Since the complement of this set is algebraic, the result follows.

For $n \geq p$ the same argument applies; we need to compute the $\mathscr{K}_e$-codimension of the homogeneous jet h. First we can use [5] to compute the Milnor number of h. Some work shows that it is

$$\mu = (-1)^{n-p+1}\left\{1+(k+1)^p\sum_{j=0}^{n-p}(-1)^{j+1}\begin{bmatrix} p+j-1 \\ j \end{bmatrix}\begin{bmatrix} n \\ n-p-j \end{bmatrix}(k+1)^j\right\}.$$

For $p = 1$ this is k^n; for $p = 2$ it is $(-1)^{n-1}\{1+(k^{n-1}(nk+n-k)(-1)^{n+1}-1)\}$; and in all cases the leading term is $\left[\begin{smallmatrix} n-1 \\ n-p \end{smallmatrix}\right]k^n$. It follows from [9, p. 509] that $d_e(h, \mathscr{K}) = \mu$ and $d_e(h+z, \mathscr{K}) \leq d_e(h, \mathscr{K})$ for any k-jet z by Lemma 1.2 again. The same argument as before shows that if $r \geq \mu + 1 (\sim \left[\begin{smallmatrix} n-1 \\ n-p \end{smallmatrix}\right]k^n)$, then all r-jets over z off a proper algebraic subset are $\mathscr{K}$-sufficient for *any* k-jet z. □

Note 1.4. It is interesting to compare our estimate for r with that of Bochnak [1; 9, p. 514]). His estimate is $r \geq 2\left[\begin{smallmatrix} (pk)^n+n-1 \\ n \end{smallmatrix}\right]$, which is truly astronomic. The leading term in k is $\frac{2}{n!}(pk)^{n^2}$, which is clearly rather larger than those obtained above (namely, pk^n for $n \leq p$ and $\left[\begin{smallmatrix} n-1 \\ n-p \end{smallmatrix}\right]k^n$ in the case $n > p$).

In [10] Wirthmüller obtains some results that also allow one to estimate r. He produces subsets $B^r \subset J^r(n, p)$ such that for $n \geq p$ the codimension of B^r is greater than or equal to $n-p+r$ ($n \leq p$, the codimension is $\geq (p-n+1)r$) and if f is a germ with $j^r f \notin B^r$ then f is $n\max\{p(r-1), r\}$-$\mathscr{K}$-determined (respectively, $\max\{nr-1, r\}$-$\mathscr{K}$-determined). This shows that given any k-jet z, most r-jets over z are $\mathscr{K}$-sufficient for $r > n\max\{p^2\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right] - pn - n, p\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right] - n\}$ (resp. $r > \max\{np(\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right]-1)/(p-n+1)-1, p\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right]/(p-n+1)\}$). Suppose $z + H \subset B^s$, where H is the space of polynomial mappings spanned by monomials of degree d with $k+1 \leq d \leq s$. Then $\operatorname{codim} B^s \leq \operatorname{codim} H = \dim J^k(n, p) = p\{\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right] - 1\}$, but $\operatorname{codim} B^s \geq n-p+s$ (resp., $(p-n+1)s$). So if $s > \{p\left[\begin{smallmatrix} n+k \\ n \end{smallmatrix}\right] - 1\}/(p-n+1))$, then there is an s-jet over z not in B_s. This, given the degrees of determinacy, implies the result.

The leading terms in these estimates are, as powers of k, $(p^2/(n-1)!)k^n$ (resp., $(p/n!(p-n+1))k^n$). So our results are weaker than those provided by [10], although of the same order. (In fact if $n \leq p$ and p is large, one can improve on the choice of h_{k+1} and bring the leading coefficient down somewhat.) We believe this order is still far too large, but now we show that one cannot improve on it using the methods above.

Example 1.5. (i) Consider the jet $z = (x_1, x_2, \ldots, x_{n-1}, 0) \in J^k(n, n)$. The homogeneous jet $h = (-x_n^{k+1}, -x_1^{k+1}, -x_2^{k+1}, \ldots, -x_{n-1}^{k+1})$ has the property by Lemma 1.2 that $z+h$ is of F.S.T. But this jet is $\mathscr{K}$-equivalent to $(x_1, \ldots, x_{n-1}, x_n^{(k+1)^n})$ and is only $(k+1)^n$-$\mathscr{K}$-determined. Of course, h is an incredibly stupid choice of a homogeneous jet to add to z to obtain a finitely determined jet, but our proof does not have the intelligence to ensure a better selection.

(ii) On the other hand one might believe that the most difficult k-jet to find $\mathscr{K}$-sufficient jets over would be the zero jet. But this has an $(nk+1)$-jet over it that is $\mathscr{K}$-sufficient in the case $n = p$; namely, $h(z) = (z_1^{k+1}, \dots, z_n^{k+1})$. Can one reduce the leading power k^n all the way down to k in the estimate for r?

2. The $\mathscr{A}$ case

Since Fact 1.1 and Lemma 1.2 hold for Mather's group $\mathscr{A}$, it is natural to ask if the results presented in §1 have analogues for this group. Sadly there are *very* few homogeneous $\mathscr{A}$-finite mapping (see Example 2.6), but in certain ranges of dimension we can find $\mathscr{A}$-finite weighted homogeneous mappings that enable us to prove that non-$\mathscr{A}$-finite maps have infinite codimension. In fact duPlessis has found all pairs of dimension (n, p) for which this is so [8]. In the cases where we can compute the codimensions of the weighted homogeneous maps, we obtain more precise information. First we list some relevant examples.

Examples 2.1. (a) $p = 1$, $f(x) = x_1^{k+1} + \cdots + x_n^{k+1}$, $d_e(f, \mathscr{A}) = k^n$.

(b) $n = 1$, $f(x) = (x^{k+1}, x^{k+2}, \dots, x^{k+p})$. If N is the integer part of $k/(p-1)$, then the $\mathscr{L}_e$ codimension of f is

$$d_e(f, \mathscr{L}) = 1/2(N+1)\{2k - N(p-1)\}p,$$

which is approximately $kp(k+p-1)/2(p-1)$. Of course $d_e(f, \mathscr{A}) \le d_e(f, \mathscr{L})$.

(c) $n = p = 2$. It follows from [5] that if $f(x, y)$ and $g(x, y)$ are generic homogeneous polynomials of degree k and $k+1$, then $(f, g)\colon \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ is finitely $\mathscr{A}$-determined and $d_e((f, g), \mathscr{A}) = 1/2\{4k^4 - 13k^2 + 7k + 2\}$.

(d) Here we briefly establish the existence of finitely $\mathscr{A}$-determined mappings $\mathbf{C}^2, 0 \to \mathbf{C}^3, 0$ that have homogeneous components of degrees (k_1, k_2, k_3), where any two of the k_i are coprime. Let $P_1, \dots, P_p$ be the spaces of homogeneous polynomials of degrees $k_1, \dots, k_p$. Set $P = P_1 \times \cdots \times P_p$, and define $F\colon \mathbf{C}^n \to P \to \mathbf{C}^p$ by

$$F(x, \phi_1, \dots, \phi_p) = (\phi_1(x), \dots, \phi_p(x)) = F_\phi(x).$$

Consider $dF\colon \mathbf{C}^n \times P \to J^1(n, p)$ defined by $dF(x, \phi) = dF_\phi(x)$.

Lemma 2.2. *The map dF is a submersion off $\{0\} \times P$.*

Proof. If $\psi \in P$ one easily checks that the image of ψ under the derivative of dF at (x, ϕ) is $d\psi(x)$. We know that $x \ne 0$, so by a linear change of coordinates we can suppose, without loss of generality, that $x = (1, 0, \dots, 0)$. Now we consider $\psi = x_1^{k_j - 1} x_i e_j$, where e_j is the jth unit vector. Clearly the space spanned by the $d\psi(x)$'s is all of $J^1(n, m)$, as required. □

Next we consider a space $\mathbf{C}^n(N)$ defined to be those N tuples $(x(1), \dots, x(N))$ of points of $\mathbf{C}^n$ with the $x(i)$ distinct and pairwise linearly independent. If $n > 1$ then this is clearly an open subset of $\mathbf{C}^{Nn}$. Consider $F(N)\colon \mathbf{C}^n(N) \times P \to (\mathbf{C}^p)^N$ defined by

$$F(N)(x, \phi) = (F_\phi(x(1)), \dots, F_\phi(x(N))).$$

Lemma 2.3. *The map $F(N)$ is transverse to the diagonal $y(1) = \cdots = y(N)$ in $(\mathbf{C}^p)^N$ if each $k_j \ge N - 1$.*

Proof. It is easy to see that if $x(1), \dots, x(N)$ are distinct and pairwise linearly independent, then for each i, j we can find a homogeneous polynomial ϕ_{ij} of degree k_j with $\phi_{ij}(x(l)) = \delta_{il}$, $1 \le j \le m$, $1 \le i, l \le N$. Now it is not difficult to see that $F(N)$ is in fact a submersion, and the result follows. □

Now we complete our examples. If $2n-1 \leq p$, then the set of singular matrices in $J^1(n,p)$ has codimension $\geq n$. Now by Thom's lemma for almost all $\phi \in P$, the map $dF_\phi\colon \mathbf{C}^n - \{0\} \to J^1(n,p)$ will be transverse to the set of singular matrices. So F_ϕ will be singular at isolated points of $\mathbb{C}^n - \{0\}$ and in particular nonsingular on some punctured neighbourhood of 0. Similarly, taking $N = 2, 3$ in the second lemma, we can see that for generic ϕ, the map $\phi\colon \mathbf{C}^n \to \mathbf{C}^p$ has the following properties:

(a) If $p \geq 2n+1$ then $\phi(x) = \phi(y)$ implies $x = \lambda y$ for some λ.
(b) If $p \geq 2n-1$ then $\phi(x) = \phi(y)$ implies either that the germs of ϕ at x and y are transverse or $x = \lambda y$ for some λ.
(c) If $p \geq 2n-1$ then $\phi(x) = \phi(y) = \phi(z)$ implies that two of x, y, or z are linearly dependent.

Now let $\phi = (\phi_1, \dots, \phi_p)$ have these properties. If $x = \lambda y$ and $\phi(x) = \phi(y)$, we have $\lambda^{k_i}\phi_i(y) = \phi_i(y)$, so either $\phi_i(y) = 0$ or $\lambda^{k_i} = 1$.

Now without loss of generality we may suppose that any n of $\phi_1, \dots, \phi_p$ have the property that the affine cones they determine meet only in $\{0\}$. So if $y \neq 0$, we see that $\phi(x) = \phi(y)$ means that $p-n+1$ of the $\phi_i(y)$ are nonzero, so $p-n+1$ of the $\lambda^{k_i} = 1$. This, together with Gaffney's characterisation of finitely $\mathscr{A}$-determined map germs [9] implies the following.

Example 2.1. (d) Let $p \geq 2n-1$, and let $k_1, \dots, k_p$ be positive integers ≥ 2 with the property that the greatest common divisor of any $p-n+1$ of them is 1. Then almost every $\phi \in P_{k_1} \times \cdots \times P_{k_p}$ is finitely $\mathscr{A}$-determined. As a particular example in the case $n = 2$, $p = 3$, if k is odd then we can consider $(k_1, k_2, k_3) = (k, k+1, k+2)$. The $\mathscr{A}_e$-codimension of this germ is, by [7], $1/6\{(k+1)^2(k^2(k+2)^2-9) + 6(k+1)(6-k(k+2)) - (3k^2+6k+21)\} = \gamma(k)$. So since f is not $\mathscr{A}$-stable, its $\mathscr{A}$-codimension is $\gamma(k)+2$ by [9].

Now we show how to use this information.

First recall that the group $\mathscr{A}$ contains a subgroup $\mathscr{A}_1$ consisting of pairs of diffeomorphisms with 1-jet the identity. Determinacy with respect to this subgroup is much easier to deal with than $\mathscr{A}$-determinacy (see [2]).

Proposition 2.4. (i) *Given (n,p), suppose that there is an $s \geq 1$ such that for each k sufficiently large, we can find a $(k+s)$-jet h_k in $\mathscr{M}_n^{k+1} \cdot \mathscr{E}(n,p)$ that is weighted homogeneous and $\mathscr{A}$-finite. Also suppose that with respect to the given weights, all jets in $J^k(n,p)$ have degrees less than that of h_k. Then $\mathscr{A}$-finiteness holds in general for smooth germs $\mathbf{R}^n, 0 \to \mathbf{R}^p, 0$.*

(ii) *Suppose further that $d(h_k, \mathscr{A}) = \alpha(k)$ for some function of k. Then almost all r-jets over any k-jet z are $\mathscr{A}$-sufficient if $r \geq \max\{k+s, 2\alpha(k)^2-1\}$. If $d(h_k, \mathscr{A}_1) = \beta(k)$ this is true for all $r \geq \max\{k+s, \beta(k)^2-1\}$.*

Proof. The proof of Proposition 2.4 is very similar to the one above. For each k define $W^{k^2+k}(n,p) = \{z \in j^{k^2+k}(n,p) : \dim V(z)/T(\mathscr{A}_1(z) + \mathscr{M}_n^{k^2+k}V(z)) \geq k+1\}$. This is clearly an algebraic set. If f is $\mathscr{A}$-finite then $T\mathscr{A}_1(f)$ contains some $\mathscr{M}_n^r \cdot V(f)$, and so $j^{s^2+s}f \notin W^{s^2+s}(n,p)$ for s sufficiently large. Conversely, if $j^{k^2+k}f \notin W^{k^2+k}(n,p)$, then by [9, 1.6(iii)] with $C = V(f)/T\mathscr{A}_1(f)$, A the image of $T\mathscr{A}_1(f)$ in C, $d = k$, we deduce that $\mathscr{M}_n^{k^2} \cdot C \subset A$, i.e., $\mathscr{M}_n^{k^2} \cdot V(f) \subset T\mathscr{A}_1(f)$. It follows from [2, 2.5] that f is (k^2-1)-$\mathscr{A}_1$-determined.

Now if we define $W^r(n, p)$ to be $(\prod_{k^2+k}^r)^{-1}(W^{k^2+k}(n, p))$, where k is the largest integer with $r \geq k^2 + k$ and $\prod_{k^2+k}^r : J^r(n, p) \to J^{k^2+k}(n, p)$ is the natural projection, we see that the nonfinitely $\mathscr{A}$-determined germs form a proalgebraic set. The argument that this set is of infinite codimension is just as before; if z is any k-jet then $z + h_k$ is $\mathscr{A}$-finite by Lemma 1.2, and this proves (i).

For the second assertion we have only to produce one r-jet over z that is $\mathscr{A}$-sufficient, but consider $f = z + h_k$ as an r-jet with $r \geq 2\alpha(k)^2 - 1$. Since $d(h_k, \mathscr{A}) = \alpha(k) \geq d(z + h_k, \mathscr{A})$, we deduce from [9, 1.6(iii)] again that $\mathscr{M}_n^{\alpha(k)^2} \cdot V(f) \subset T\mathscr{A}(f)$, and so [9, 1.2] shows that f is $(2\alpha(k)^2 - 1)$-$\mathscr{A}$-determined.

One obtains better estimates for r in some cases by considering $d(h_k, \mathscr{A}_1) = \beta(k)$. The argument above, but replacing [9, 1.2] with [2,2.5], shows that we may take r to be any integer $\geq \beta(k)^2 - 1$. □

Example 2.5. So if $z: \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ is any $(k-1)$-jet, then almost all r-jets over z are $\mathscr{A}$-sufficient if $r \geq 2(\alpha(k))^2 - 1$, where $\alpha(k) = 1/2\{4k^4 - 13k^2 + 7k + 2\} + 2$.

The search for homogeneous $\mathscr{A}$-finite map-germs led to the following.

Example 2.6. Suppose that (n, p) is the nice dimensions, $n \leq p$. Then for all k we can find a homogeneous map germ $h: \mathbf{R}^{n+1}, 0 \to \mathbf{R}^{p+1}, 0$ of degree $2k+1$ that is $\mathscr{A}$-semifinite but not $\mathscr{A}$-finite (here we are using the terminology of [9, Part II]). Indeed almost all homogeneous germs h have this property.

Proof. Consider the Veronese embedding $V: \mathbf{R}P^n \to \mathbf{R}P^N$ determined by the N homogeneous polynomials of degree $2k+1$ in $x_0, \ldots, x_n$. By [6] a generic projection of $V(\mathbf{R}P^n)$ to $\mathbf{R}P^p$ will be stable (and well defined, since $n \leq p$). But this projection corresponds to a homogeneous map $\overline{h}: \mathbf{R}P^n \to \mathbf{R}P^p$; let $h: \mathbf{R}^{n+1}, 0 \to \mathbf{R}^{p+1}, 0$ be the corresponding germ. We claim that h is $\mathscr{K}$-finite and for every finite subset $S \subset \mathbf{R}^{n+1} - \{0\}$, the multigerm of h at S is $\mathscr{A}$-stable (which one can take as the definition of $\mathscr{A}$-semifinite (see [9, §§8, 9])). To prove the claim, note that if $h(x) = h(y)$ and $x = \lambda y$, then $x = y$ for $h(\lambda y) = \lambda^{2k+1}h(y)$. Now it is easy to see that the multigerms are trivial unfoldings of $\mathscr{A}$-stable multigerms and hence $\mathscr{A}$-stable. Any generic h yields a complexification $h_{\mathbf{C}}: \mathbf{C}^{n+1}, 0 \to \mathbf{C}^{p+1}, 0$ that is finite, and so h is $\mathscr{K}$-finite too. We are done.

On the other hand no homogeneous germ $h: \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ of degree $k > \max(n, p) + 1$ is $\mathscr{A}$-finite. For if ω is any kth root of unity, then $h(\omega x) = h(x)$. If $n < p$ a stable multigerm $\mathbf{R}^{n+1}, S \to \mathbf{R}^{p+1}, 0$ has $|S| \leq p+1$, so if $k > p+1$ the germ h is not $\mathscr{A}$-finite. If $n \geq p$ and $k > 1$, then $h_{\mathbf{C}}: \mathbf{C}^{n+1}, 0 \to \mathbf{C}^{p+1}, 0$ has a nonempty homogeneous critical set Σ given by the vanishing of the $(p+1) \times (p+1)$ minors of the Jacobian matrix $[\partial h_i/\partial x_j]$. If $\overline{x} \in \Sigma$ then clearly $t\omega\overline{x} \in \Sigma$ for ω any kth root of unity, t any complex number. So we can find, in any neighbourhood U of $\mathbf{C}^n - \{0\}$, a multigerm $h_{\mathbf{C}}: \mathbf{C}^{n+1}, S \to \mathbf{C}^{p+1}, y$ with $|S| \geq k$ and $h_{\mathbf{C}}$ singular at each point x of S. If $k > n+1$ again we deduce that the multigerm is not $\mathscr{A}$-stable, and so $h_{\mathbf{C}}$ is not $\mathscr{A}$-finite. □

Note 2.7. The same argument shows that if we assign positive integral weights

w_i to x_i and $f: \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ is weighted homogeneous of degree k (i.e., $\deg f_1 = \cdots = \deg f_p = k$), then f is not $\mathscr{A}$-finite if, setting $\alpha_i = \gcd(k, w_i)$, we have $(k/\max\{\alpha_i\}) > \max(n, p) + 1$. For if ω is a kth root of unity and we set $\omega \cdot x = (\omega^{w_1} x_1, \ldots, \omega^{w_n} x_n)$, then $f(\omega \cdot x) = \omega^k \cdot f(x) = f(x)$. If $x \neq 0$ then the set $\{\omega \cdot x : \omega^k = 1\}$ has at least $k/\max\{\alpha_i\}$ elements, and the proof finishes as above.

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