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BOOTSTRAPPING REGULARITY OF THE ANOSOV SPLITTING

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ABSTRACT. Finite smoothness of the Anosov splitting implies C^∞.

Definitions. Let M be a compact Riemannian manifold. $f \in \mathrm{Diff}^\infty(M)$ is called Anosov with Anosov splitting $TM = E^u \oplus E^s$ if

$$(\exists C\,,\ \epsilon > 0)\,,$$
$$(\forall p \in M)\ \ (\exists \mu_1 < \mu_2 < 1-\epsilon < 1+\epsilon < \nu_2 < \nu_1)$$
$$(\forall v \in E^s(p)\,,\ u \in E^u(p)\,,\ n \in \mathbb{N})\,,$$
$$(\mu_1^n/C)\|v\| \le \|Df^n(v)\| \le C\mu_2^n\|v\| \text{ and}$$
$$(\nu_1^{-n}/C)\|u\| \le \|Df^{-n}(u)\| \le C\nu_2^{-n}\|u\|\,.$$

f is called α-bunched if $\sup_{p\in M} \mu_2\nu_2^{-1}(\min(\mu_1\,,\ \nu_1^{-1}))^{-\alpha} < 1$.
Riemannian metrics with C^∞-conjugate geodesic flows are called *isodynamic.*

Theorem. *If* $\sup_{p\in M} \nu_1\mu_1^{-1}\mu_2^n < 1$ *and* $E^u \in C^n$ *then* $E^u \in C^\infty$.
The same holds for the weak unstable distribution of Anosov flows.

Katok obtained a similar result via nonstationary normal forms (oral communication). Since our technique is standard [HK, LMM], we only sketch the proof. We first list corollaries. Considering f^{-1} yields

Corollary 1. *If* $\sup_{p\in M} \nu_1\mu_1^{-1}\nu_2^{-n} < 1$ *and* $E^s \in C^n$ *then* $E^s \in C^\infty$.

Corollary 2. *If* f *preserves volume,* codim $E^u = 1$, *and* $E^u \in C^2$, *then* $E^u \in C^\infty$.

Remark. Ghys [G1] proves that a volume preserving Anosov flow on a compact manifold of dimension greater than three is a suspension if $\operatorname{codim} E^u = 1$ and $E^u \in C^2$. Corollary 2 suggests that his result might be a rigidity statement.

Corollary 3. *If* $\dim M = 2$ *and* $-\log\|Df^k\restriction_{E^u(p)}\|/\log\|Df^k\restriction_{E^s(p)}\| < n-1-\delta$ *for all* $k \in \mathbb{N}$, $p \in M$, *and* $E^u \in C^n$, *then* $E^u \in C^\infty$.

Remark. Thus Theorem C of Ghys [G2] about the "generalized algebraicity" of flows with C^∞ Anosov splitting holds with a C^3-assumption.

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Corollary 4. *If $n > 2$ then $\frac{4}{n}$-bunched symplectic Anosov systems with C^n Anosov splitting have C^∞ Anosov splitting.*

Proof. For symplectic Anosov systems $\mu_i^{-1} = \nu_i$. □

Fact. [Kl] A γ-pinched Riemannian metric (where $\Lambda <$ sectional curvature $\leq \gamma\Lambda$ for some $\Lambda < 0$) has $2\sqrt{\gamma}$-bunched Anosov geodesic flow.

Corollary 5. *Kanai's theorem* [K] *holds with a C^3 hypothesis.*

Corollary 6. *If $n > 2$ then $\frac{4}{n^2}$-pinched Riemannian metrics with C^n horospheric foliations are isodynamic to a locally symmetric metric.*

Proof. By Corollary 4 this follows from the fact [BFL, F, FK, K] that metrics with C^∞ horospheric foliations are isodynamic to a locally symmetric metric. □

Proof. E^u, E^s are tangent to foliations W^u, W^s with smooth leaves. Around $p \in M$ introduce local C^∞ coordinates $\psi_p\colon q \mapsto (x, y) \in \mathbb{R}^j \oplus \mathbb{R}^l$ (continuous in p in the C^∞-topology) so that $q \in W^u(p) \Leftrightarrow \psi_p(q) = (x, 0)$ and $q \in W^s(p) \Leftrightarrow \psi_p(q) = (0, y) =: y_0 \in \{0\} \oplus \mathbb{R}^l$ (for these coordinates and for flows refer to [H]). Identify $E^u(q)$ via $D_q\psi_p$ with an affine subspace $E^u(y_0)$ of $\mathbb{R}^j \oplus \mathbb{R}^l$ containing y_0. Since $E^u(0) = \mathbb{R}^j \oplus \{0\}$ and by continuity, $E^u(y_0)$ is, after parallel translation, the graph of a linear map $E(y_0)\colon \mathbb{R}^j \to \mathbb{R}^l$ or the image of $\begin{pmatrix} I \\ E(y_0) \end{pmatrix}\colon \mathbb{R}^j \to \mathbb{R}^j \oplus \mathbb{R}^l$, where I is the identity of $\mathbb{R}^j$. $E^u(fq)$ is, via ψ_{fp}, the image of $\begin{pmatrix} I \\ E(y_1) \end{pmatrix}\colon \mathbb{R}^j \to \mathbb{R}^j \oplus \mathbb{R}^l$, which since $E^u(fq) = D_q f E^u(q)$ and $D_q f$ is (by invariance of $W^s(p)$) lower block triangular in local coordinates, coincides with that of

$$D_q f \begin{pmatrix} I \\ E(y_0) \end{pmatrix} = \begin{pmatrix} a_{y_0} & 0 \\ b_{y_0} & c_{y_0} \end{pmatrix} \begin{pmatrix} I \\ E(y_0) \end{pmatrix} = \begin{pmatrix} a_{y_0} \\ b_{y_0} + c_{y_0} E(y_0) \end{pmatrix};$$

whence, by reparametrizing the preimage by $a_{y_0}^{-1}$,

$$E(y_1) = c_{y_0} E(y_0) a_{y_0}^{-1} + b_{y_0} a_{y_0}^{-1}. \tag{1}$$

Denote by C_s^k the space of functions that are uniformly C^k along stable leaves. We now show that if $k \geq n$, $E \in C_s^k$, and $\sup_{p\in M} \nu_1 \mu_1^{-1} \mu_2^n < 1$, then $E(\cdot) \in C_s^{k+1}$. Differentiating (1) k times yields

$$(E(y_1))^{(k)} = (c_{y_0} E_{y_0} a_{y_0}^{-1})^{(k)} + (b_{y_0} a_{y_0}^{-1})^{(k)}$$

or, with $\tilde{f} := f\restriction_{W^s(p)}\colon W^s(p) \to W^s(p)$ and $F_p^i := \prod_{l=1}^i (D\tilde{f}(f^l(p)))^k$,

$$D_s^k E(f(p)) F_p^1 = c_p D_s^k E(p) a_p^{-1} - \zeta(p)$$

where $\zeta \in C^1$ since ζ is a polynomial in derivatives of E up to order $k-1$ and those of f. Thus

$$D_s^k E(p) = c_p^{-1} D_s^k E(f(p)) F_p^1 a_p + c_p^{-1} \zeta_p a_p$$

and recursively, with $A_p^i := \prod_{l=0}^i a_{f^{i-l}p}$ and $C_p^i := \prod_{l=0}^i c_{f^l p}^{-1}$,

$$D_s^k E(p) = C_p^{m-1} D_s^k E(f^m(p)) F_p^m A_p^{m-1} + \sum_{i=0}^{m-1} C_p^i \zeta_{f^i p} F_p^{i-1} A_p^i.$$

Since $\|C_p^{m-1}\| \leq C\mu_1^{-m}$, $\|A_p^{m-1}\| \leq C\nu_1^m$ and $\|F_p^m\| \leq C\mu_2^{km}$, we have

$$D_s^k E(p) = \sum_{i=0}^{\infty} C_p^i \zeta_{f^i p} F_p^{i-1} A_p^i.$$

$\|D_z C_p^{i-1}\| \leq Ci\mu_1^{-i}$, $\|D_z A_p^{i-1}\| \leq Ci\nu_1^i$, and $\|D_z F_p^{i-1}\| \leq Ci\mu_2^{ki}$ by the product rule, so derivatives of the terms of the above series have a convergent sum (uniformly in p by exponential convergence) and $E \in C_s^{k+1}$. Inductively $E \in C_s^\infty$. By a result of Journé [J] the fact that W^s and W^u are two continuous transverse foliations with uniformly smooth leaves implies that $C_s^\infty \cap C_u^\infty = C^\infty$. Thus $E \in C^\infty$. □

Proof of Corollary 2. If $\dim M = 2$ see [HK], otherwise $\|C_p^i F_p^{i-1} A_p^i\| < Ce^{-\epsilon i}$ and $D_z\|C_p^i F_p^{i-1} A_p^i\| < Cie^{-\epsilon i}$ for $k \geq 2$, $p \in M$, and the above argument applies. □

Proof of Corollary 3. $\|C_p^i F_p^{i-1} A_p^i\| < Ce^{-\epsilon i}$ and $D_z\|C_p^i F_p^{i-1} A_p^i\| < Cie^{-\epsilon i}$ for $k > n-1$, $p \in M$, and appropriate $\epsilon > 0$. □

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References

[BFL] Yves Benoist, Patrick Foulon, and François Labourie, *Flots d'Anosov a distributions stable et instable differentiables*, École Polytechnique preprint M949.0690, 1990.

[F] Renato Feres, *Geodesic flows on Manifolds of Negative Curvature with Smooth Horospherical Foliations*, Ergodic Theory Dynamical Systems **11** (1991), no. 4, 653–686.

[FK] Renato Feres and Anatoly B. Katok, *Invariant tensor fields of dynamical systems with pinched Lyapunov exponents and rigidity of geodesic flows*, Ergodic Theory and Dynamical Systems **9** (1989), 427–432; *Anosov flows with smooth foliations and rigidity of geodesic flows in three-dimensional manifolds of negative curvature*, Ergodic Theory and Dynamical Systems **10** (1990), 657–670.

[G1] Etienne Ghys, *Codimension one Anosov flows and suspensions*, Dynamical Systems Valparaiso 1986, (Rodrigo Bamón, Rafael Labarea, and Jacob Palis Jr., eds.), Lecture Notes in Mathematics vol. 1331, Springer-Verlag, Berlin, Heidelberg, and New York, 1988, pp. 59–72.

[G2] Etienne Ghys, *Déformations de flots d'Anosov et de groupes Fuchsiens*, Annales de l'Institut Fourier (1992) (to appear).

[H] Borís Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, preprint IHES/M/91/1, 1991.

[HK] Steven Hurder and Anatoly B. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, Publications Mathematiques de l'IHES **72** (1990), 5–61.

[J] Jean-Lin Journé, *A regularity lemma for functions of several variables*, Rev. Mat. Ibero. **4** (1988), 187–193.

[K] Masahiko Kanai, *Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations*, Ergodic Theory and Dynamical Systems **8** 2 (1988), 215–240.

[Kl] Wilhelm Klingenberg, *Riemannian geometry*, Studies in Math., de Gruyter, 1982, p. 276.

[LMM] Rafael de la Llave, J. M. Marco, and R. Moriyon, *Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation*, Ann. of Math. **123** (1986), 537–611.

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