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ON THE WEIGHTED L^p-INTEGRABILITY OF NONNEGATIVE $\mathcal{M}$-SUPERHARMONIC FUNCTIONS

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ABSTRACT. A weighted L^p-integrability of nonnegative $\mathcal{M}$-superharmonic functions in the unit ball of $\mathbb{C}^n$ is studied in this paper. Our result is analogous to an earlier result of Armitage (J. London Math. Soc. (2) **4** (1971), 363–373) concerning the L^p-integrability of superharmonic functions for balls in $\mathbb{R}^d$. An example is given to show the sharpness of the result. Also, the weighted L^p-integrability of the invariant Green's function for the unit ball of $\mathbb{C}^n$ is obtained.

1. INTRODUCTION

In [2], Armitage studied the global integrability of superharmonic functions in balls of $\mathbb{R}^d$, and he proved the following

Theorem A. *If s is positive and superharmonic in $B(0, 1) \subseteq \mathbb{R}^d$, $d \geq 2$, and $0 < p < d/(d-1)$, then $s \in L^p(B(0, 1))$, and*

$$\int_{B(0,1)} s^p(x)\, dx \leq A(d, p) s^p(0), \tag{1.1}$$

where $A(d, p)$ is a positive constant depending only on d and p.

The purpose of this paper is to prove the analogue of Theorem A on the unit ball B of $\mathbb{C}^n$. Our result is as follows (for the notation used in this section see §2).

Theorem 1. *Let u be a nonnegative $\mathcal{M}$-superharmonic function in the unit ball B of $\mathbb{C}^n$, $0 < a \leq n/(n-1)$, and $0 < p < 1 + \alpha/n$. Then, for each $a \in B$ there exists a constant $A(n, \alpha, p, a)$, independent of u, such that*

$$\int_B (1 - |z|^2)^{\alpha - 1} u^p(z)\, d\nu(z) \leq A(n, \alpha, p, a) u^p(a). \tag{1.2}$$

In particular, u is L^p-integrable on B with respect to the measure

$$(1 - |z|^2)^{\alpha - 1}\, d\nu(z).$$

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Here ν denotes the normalized Lebesgue measure on B so that $\nu(B) = 1$.

The unweighted case, i.e., when $\alpha = 1$, is most interesting for us. In this case, we obtain the L^p-integrability $(0 < p < (n+1)/n)$ of nonnegative $\mathcal{M}$-superharmonic functions with respect to the Lebesgue measure ν on B. Comparing $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the upper bound of p in our case is $(n+1)/n$, which is strictly greater than $2n/(2n-1)$, the upper bound of p in Theorem A if $n > 1$. Of course, for $n = 1$, these two upper bounds coincide.

Using a similar method of proof as with Theorem 1, we obtain the following result on the integrability of invariant Green's function on B.

Theorem 2. *Let G denote the invariant Green's function on B. Let $\alpha > -n^2/(n-1)$ and $p > 0$. Then there exists a constant $A(n, p, \alpha)$ such that*

$$\sup_{w \in B} \int_B (1 - |z|^2)^{\alpha - 1} G^p(z, w)\, d\nu(z) \le A(n, p, \alpha) \tag{1.3}$$

if and only if $-\alpha/n < p < n/(n-1)$.

We remark that it has been observed in [3] that the invariant Green's function G is L^p-integrable with respect to the invariant measure $d\tau(z) := (1 - |z|^2)^{-n-1}\, d\nu(z)$ on B for $1 < p < n/(n-1)$. This is a special case of the last theorem with $\alpha = -n$. Another interesting case is when $\alpha = 1$. In this case, Theorem 2 asserts, in particular, that the invariant Green's function G is integrable with respect to the Lebesgue measure ν on B.

For the purpose of comparison, we would like to point out that it was proved in [4] that, for a class of uniformly elliptic PDE's Green's functions of bounded domains in $\mathbb{R}^d$ $(d \ge 1)$ are L^p-integrable for some $p > d/(d-1)$. In our concrete case, the invariant Laplace equation in B is no longer uniformly elliptic (see [7, Theorem 4.1.3(ii)]). Nevertheless, we still have the best possible result in this situation.

In §2 we explain the notation in this paper and recall a few preliminary facts of potential theory in the unit ball of $\mathbb{C}^n$ that are needed in the sequel. In §3 we state several lemmas concerning estimates of some integrals and the invariant Green's function on the unit ball of $\mathbb{C}^n$. Proofs of these lemmas can either be found in the references or follow easily from results that appear there. Based on these estimates, we prove Theorems 1 and 2 in §4. We will follow the idea of Armitage [2], but some different arguments will be used. Finally, in §5, we give an example to show the sharpness of the upper bound of p in Theorem 1.

2. Preliminaries

Throughout this paper, n denotes a positive integer, and we will assume that $n > 1$. Our results will coincide with the one in $\mathbb{R}^{2n}$ in the case of $n = 1$, as we mentioned before.

For $z, w \in \mathbb{C}^n$, set $\langle z, w \rangle := \sum_{j=1}^n z_j \overline{w}_j$ and $|z|^2 := \langle z, z \rangle$. For $\delta > 0$, let $B_\delta := \{z \in \mathbb{C}^n : |z| < \delta\}$ and $S_\delta := \{z \in \mathbb{C}^n : |z| = \delta\}$. The unit ball of $\mathbb{C}^n$ is then $B := B_1$, and the unit sphere of $\mathbb{C}^n$ is $S := S_1$.

For each $a \in B$, let ϕ_a denote the involutive automorphism of B for which $\phi_a(0) = a$ and $\phi_a \circ \phi_a(z) = z$. We then have the identity [7, p. 26]

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle a, z \rangle|^2}. \tag{2.1}$$

For a given δ, $1 \le \delta \le 1$, and $a \in B$, we set $E(a, \delta) := \{z \in B : |\phi_a(z)| < \delta\} = \phi_a(B_\delta)$. Then we have the change of variables formula,

$$\int_{E(a,\delta)} f(w)\,d\nu(w) = \int_{B_\delta} f(\phi_a(z)) \left(\frac{1-|a|^2}{|1-\langle a, z\rangle|^2}\right)^{n+1} d\nu(z) \tag{2.2}$$

for every integrable function f on B. This follows from the usual change of variables formula and the form of the real Jacobian of ϕ_a at $z \in B$ [7, p. 28].

Now we recall that if $\mathcal{M}$ denote the group of holomorphic automorphisms of B, then each $\psi \in \mathcal{M}$ has a unique representation $\psi = U \circ \phi_a$ for some $a \in B$ and $U \in \mathbf{U}(n)$, where $\mathbf{U}(n)$ denotes the group of unitary transformations of $\mathbb{C}^n$.

A lower semicontinuous function $u : B \to (-\infty, \infty]$ is said to be *$\mathcal{M}$-superharmonic* in B if $u \not\equiv \infty$ and it has the invariant-super-mean-value property

$$u(a) \ge \int_S u(\phi_a(r\zeta))\,d\sigma(\zeta) \tag{2.3}$$

for all $a \in B$ and $0 < r < 1$. Here σ denote the rotation-invariant measure on S normalized so that $\sigma(S) = 1$. A function v is said to be *$\mathcal{M}$-subharmonic* if $-v$ is $\mathcal{M}$-superharmonic, and a function h is said to be *$\mathcal{M}$-harmonic* if it is both $\mathcal{M}$-superharmonic and $\mathcal{M}$-subharmonic.

From the definitions, it is easy to see that $\mathcal{M}$-harmonic and $\mathcal{M}$-superharmonic functions are $\mathcal{M}$-invariant, e.g., if u is $\mathcal{M}$-superharmonic then $u \circ \psi$ is also $\mathcal{M}$-superharmonic for any $\psi \in \mathcal{M}$.

The *invariant Poisson kernel* on B is given by

$$P(z, \zeta) := \frac{(1-|z|^2)^n}{|1-\langle z, \zeta\rangle|^{2n}}, \qquad z \in B,\ \zeta \in S. \tag{2.4}$$

For a measure ω on S, we write

$$P\omega(z) := \int_S P(z, \zeta)\,d\omega(\zeta). \tag{2.5}$$

The *invariant Green's function* on B is given by

$$G(z, w) := g(\phi_z(w)), \qquad z, w \in B, \tag{2.6}$$

where

$$g(z) := c(n) \int_{|z|}^1 \frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt \tag{2.7}$$

for an appropriate positive constant $c(n)$ depending only on n.

A nonnegative $\mathcal{M}$-superharmonic function on B is called an *invariant potential* if it has no positive $\mathcal{M}$-harmonic minorant. As a consequence of Theorem C, the invariant potentials are precisely the functions of the form

$$G\mu(z) := \int_B G(z, w)\,d\mu(w) \tag{2.8}$$

for some nonnegative measure μ on B so that $G\mu \not\equiv \infty$, or equivalently, μ satisfies the condition

$$\int_B (1-|w|^2)^n\,d\mu(w) < \infty. \tag{2.9}$$

The following characterizations of $\mathcal{M}$-harmonic and $\mathcal{M}$-superharmonic functions are well known.

Theorem B [7, Theorem 4.3.3.]. *Suppose h is $\mathcal{M}$-harmonic in B. There exists a finite measure ω on S such that $h = P\omega$ if and only if*

$$\sup_{1/2\le r<1} \int_S |h(r\zeta)|\, d\sigma(\zeta) < \infty. \tag{2.10}$$

Theorem C [9, Theorem 2.16]. *If μ is $\mathcal{M}$-superharmonic in B and satisfies growth condition* (2.10), *then $u = h + G\mu$, where h is the greatest $\mathcal{M}$-harmonic minorant of u and μ is a nonnegative measure on B satisfying condition* (2.9).

3. Some lemmas

In what follows, we use $A(\alpha, \beta, \dots)$ or C to denote a positive constant depending only on the constant $\alpha, \beta, \dots$, not necessarily the same on any two occurrences.

The proof of the following lemma is almost identical to [7, Proposition 1.4.10] and consequently is omitted.

Lemma 3. *Let p, α, and δ be real numbers.*

(1) *If $0 < \delta < 1$ and $\alpha > -n$, then there exists a constant $A(n, \alpha, \delta)$ such that*

$$I(n, \alpha, r, w) := \int_S \frac{1}{|1 - r\langle\zeta, w\rangle|^{2(n+\alpha)}}\, d\sigma(\zeta) \le A(n, \alpha, \delta) \tag{3.1}$$

for all $w \in B$ and $0 < r \le \delta$.

(2) *If $p > 0$ and $\alpha > -n/2$, then there exists a constant $A(n, \alpha, p)$ such that*

$$J(n, \alpha, p, w) := \int_B \frac{(1-|z|^2)^{np+\alpha-1}}{|1 - \langle z, w\rangle|^{2np}}\, d\nu(z) \le A(n, \alpha, p) \tag{3.2}$$

for all $w \in \overline{B}$ if and only if $-\alpha/n < p < 1 + \alpha/n$.

(3) *If $p > 0$ and $\alpha > -np$, then there is a constant $A(n, \alpha, p)$ such that*

$$\sup_{w\in B}\{1 - |w|^2)^{np} J(n, \alpha, p, w)\} \le A(n, \alpha, p). \tag{3.3}$$

The following estimates were observed in [9, 8].

Lemma 4. *Let g be as defined by* (2.7) *and $0 < \delta < 1$. Then there exist constants C_1 and C_2 depending only on n and δ such that*

$$g(w) \ge C_1(1 - |w|^2)^n \quad \text{for all } w \in B; \tag{3.4}$$

$$g(w) \le C_2(1 - |w|^2)^n \quad \text{for all } w \in B\backslash B_\delta; \tag{3.5}$$

$$C_1|w|^{-2(n-1)} \le g(w) \le C_2|w|^{-2(n-1)} \quad \text{for all } w \in \overline{B_\delta}. \tag{3.6}$$

The next lemma is an immediate consequence of Lemma 4 and the definition of the invariant Green's function on B.

Lemma 5. *For the invariant Green's function G, there exists a constant $A(n, \delta)$ such that*

$$G(z, w) \le A(n, \delta) G(0, w) \frac{(1-|z|^2)^n}{|1 - \langle z, w\rangle|^{2n}} \tag{3.7}$$

for all $w \in B$, $z \in B \setminus E(w, \delta)$, *and*

$$G(z, w) \leq A(n, \delta) G(0, w) \frac{1}{|\phi_z(w)|^{2(n-1)}(1 - |w|^2)^n} \tag{3.8}$$

for all $w \in B$, $z \in E(w, \delta)$.

4. Proofs of the results

Before we start to prove Theorem 1, we remark that the positivity of the $\mathscr{M}$-superharmonic function u in the statement of Theorem 1 can be replaced by a weaker, but somewhat technical hypothesis; namely, that u satisfies growth condition (2.10). The latter always holds when our simple hypothesis is satisfied.

Now we begin with the special case of Theorem 1, where the function u is $\mathscr{M}$-harmonic.

Proposition 6. *If* h *is a nonnegative* $\mathscr{M}$*-harmonic in* B, $\alpha > 0$, $0 < p < 1 + \alpha/n$, *then there exists a constant* $A(n, \alpha, p)$ *such that*

$$\int_B (1 - |z|^2)^{\alpha - 1} h^p(z)\, d\nu(z) \leq A(n, \alpha, p) h^p(0) < \infty. \tag{4.1}$$

Proof. By Theorem B, there exists a nonnegative finite measure ω on S such that $h = P\omega$, since $h \geq 0$. Notice that $P(0, \zeta) = 1$ for all $\zeta \in S$, so that

$$h(0) \int_S d\omega(\zeta). \tag{4.2}$$

If $1 \leq p < 1 + \alpha/n$, then Hölder's inequality, (4.2), and inequality (3.2) imply that

$$\begin{aligned}
&\int_B (1 - |z|^2)^{\alpha-1} h^p(z)\, d\nu(z) \\
&\quad = \int_B (1 - |z|^2)^{\alpha-1} \left(\int_S P(z, \zeta)\, d\omega(\zeta) \right)^p d\nu(z) \\
&\quad \leq \int_B (1 - |z|^2)^{\alpha-1} \left\{ \left(\int_S d\omega(\zeta) \right)^{p-1} \int_S P^p(z, \zeta)\, d\omega(\zeta) \right\} d\nu(z) \\
&\quad = h^{p-1}(0) \int_S J(n, \alpha, p, \zeta)\, d\omega(\zeta) \\
&\quad \leq A(n, \alpha, p) h^{p-1}(0) \int_S d\omega(\zeta) = A(n, \alpha, p) h^p(0).
\end{aligned}$$

The change of order of integration is justified by Fubini's theorem, since P is positive.

For the case of $0 < p < 1$, we apply Hölder's inequality with respect to the measure $(1 - |z|^2)^{\alpha-1}\, d\nu(z)$ to get

$$\begin{aligned}
&\int_B (1 - |z|^2)^{\alpha-1} h^p(z)\, d\nu(z) \\
&\quad \leq \left(\int_B (1 - |z|^2)^{\alpha-1} h(z)\, d\nu(z) \right)^p \left(\int_B (1 - |z|^2)^{\alpha-1}\, d\nu(z) \right)^{1-p} \\
&\quad \leq A(n, \alpha, p) h^p(0). \quad \square
\end{aligned}$$

Since every $\mathcal{M}$-subharmonic function that satisfies growth condition (2.10) has a least $\mathcal{M}$-harmonic majorant by Theorem C, the following corollary is an easy consequence of Proposition 6.

Corollary 7. *If v is a nonnegative $\mathcal{M}$-subharmonic function satisfying growth condition* (2.10), *$\alpha > 0$, and $0 < p < 1 + \alpha/n$, then*

$$\int_B (1-|z|^2)^{\alpha-1} v^p(z)\, d\nu(z) < \infty. \tag{4.3}$$

Now we give the key estimate for the invariant Green's function, which leads us to the integrability of nonnegative $\mathcal{M}$-superharmonic functions.

Proposition 8. *If $-n/2 < \alpha \le n/(n-1)$ and $\max\{0, -\alpha/n\} < p < 1 + \alpha/n$, then there exists a constant $A(n, \alpha, p)$ such that*

$$\int_B (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) \le A(n, \alpha, p) G^p(0, w) \tag{4.4}$$

for all $w \in B$.

Proof. Choose $\delta = \frac{1}{2}$, and fix $w \in B$. We divide the integral on the left side of (4.4) into two parts

$$\int_B (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) = \int_{B\setminus E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) + \int_{E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z).$$

By using (3.7) and (3.2), we have

$$\begin{aligned}\int_{B\setminus E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) &\le A(n) G^p(0, w) J(n, p, \alpha, w)\\ &\le A(n, \alpha, p) G^p(0, w).\end{aligned}$$

Now, (3.8), (2.1), (2.2), and (3.1) imply that

$$\begin{aligned}&\int_{E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z)\\ &\quad\le \frac{A(n) G^p(0, w)}{(1-|w|^2)^{np}} \int_{E(w,\delta)} \frac{(1-|z|^2)^{\alpha-1}}{|\phi_w(z)|^{2(n-1)p}}\, d\nu(z)\\ &\quad= \frac{A(n) G^p(0, w)}{(1-|w|^2)^{np}} \int_{B_\delta} \frac{(1-|\phi_w(z)|^2)^{\alpha-1}}{|z|^{2(n-1)p}} \left(\frac{1-|w|^2}{|1-\langle w, z\rangle|^2}\right)^{n+1} d\nu(z)\\ &\quad= A(n) G^p(0, w)(1-|w|^2)^{n+\alpha-np} \int_{B_\delta} \frac{(1-|z|^2)^{\alpha-1}}{|z|^{2(n-1)p}|1-\langle w, z\rangle|^{2(n+\alpha)}}\, d\nu(z)\\ &\quad= A(n) G^p(0, w) \int_0^\delta I(n, \alpha, r, w) \frac{r^{2n-1}(1-r^2)^{\alpha-1}}{r^{2(n-1)p}}\, dr\\ &\quad\le A(n, \alpha, p) G^p(0, w),\end{aligned}$$

where integration in polar coordinates has been used. □

The last proposition has an interesting corollary, which follows easily from Hölder's inequality.

Corollary 9. *If f is a nonnegative and $L^q(\nu)$-integrable function in B, $q > n+1$, then*

$$\lim_{|z|\to 1} Gf(z) = 0. \tag{4.5}$$

Here $Gf(z) := \int_B G(z, w)f(w)\,d\nu(w)$.

Next, we look at the special case of Theorem 1 when u is an invariant potential.

Proposition 10. *If $G\mu$ is an invariant potential, $0 < \alpha \le n/(n-1)$, and $0 < p < 1 + \alpha/n$, then there exists a positive constant $A(n, \alpha, p)$ such that*

$$\int_B (1-|z|^2)^{\alpha-1}(G\mu)^p(z)\,d\nu(z) \le A(n, \alpha, p)(G\mu)^p(0). \tag{4.6}$$

Proof. If $1 \le p < 1 + \alpha/n$, then this follows from Proposition 8 by the continuous version of Minkowski's inequality with respect to the measure $(1-|z|^2)^{\alpha-1}\,d\nu(z)$. The general case then follows by applying Hölder's inequality, as in Proposition 6. □

Proof of Theorem 1. The second assertion follows from the first since the set $\{a \in B\colon u(a) = \infty\}$ has zero ν-measure (see [9, Corollary 2.17]). Also, as we noticed before, it is enough to prove (1.2) in the case of $1 \le p < 1 + \alpha/n$.

As a consequence of Theorem C, there exist nonnegative measures ω on S and μ on B such that $u = P\omega + G\mu$. Therefore, (1.2) follows from Propositions 6 and 8 for $a = o$, by an application of Minkowski's inequality for the L^p-norm with respect to the measure $(1-|z|(2)^{\alpha-1}\,d\nu(z)$.

For the case where $a \ne 0$, we use the invariant property of $\mathcal{M}$-superharmonic functions. Since the function $u \circ \phi_a$ is also $\mathcal{M}$-superharmonic in B, we are able to apply the previous result to $u \circ \phi_a$. Now, by using the facts that $\phi_a \circ \phi_a$ is the identity map and $\phi_a(0) = a$, we have

$$\begin{aligned}
&\int_B (1-|z|(2)^{\alpha-1}u^p(z)\,d\nu(z)\\
&= \int_B (1-|\phi_a(\phi_a(z))|^2)^{\alpha-1}(u\circ\phi_a(z))\,d\nu(z)\\
&= \int_B (1-|\phi_a(z)|^2)^{\alpha-1}(u\circ\phi_a)p(z)\left(\frac{1|z|^2}{|1-\langle a, z\rangle|^2}\right)^{n+1} d\nu(z)\\
&= (1-|a|^2)^{n+\alpha}\int_B (u\circ\phi_a)^p(z)\frac{(1-|z|^2)^{\alpha-1}}{|1-\langle a, z\rangle|^{2(n+\alpha)}}\,d\nu(z)\\
&\le A(n, \alpha, p, a)u^p(z).
\end{aligned}$$

Here we have applied identity (2.1) and the estimate $|1-\langle a, z\rangle|^{-2(n+\alpha)} \le A(n, \alpha, a)$ for $z \in B$. Thus, the theorem is established. □

Proof of Theorem 2. Estimate (3.6) shows that the integral on the left-hand side of (1.3) is infinite if $p \ge n/(n-1)$, similarly if $p \le -\alpha/n$ by (3.4).

On the other hand, for fixed $w \in B$ and $\delta = \frac{1}{2}$, estimates (3.5) and (3.6) give us

$$G(z, w) \le A(n)\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z, w\rangle|^2}\right)^n$$

for all $z \in B \backslash E(w, \delta)$ and

$$G(w, z) \leq A(n) \frac{1}{|\phi_z(w)|^{2(n-1)}}$$

for all $z \in E(w, \delta)$. The same computation as in Proposition 6 shows that, for $-\alpha/n < p < n/(n-1)$,

$$\begin{aligned}
&\int_B (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) \\
&\quad = \int_{B\backslash E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) \\
&\qquad + \int_{E(w,\delta)} (1-|z|^2)^{\alpha-1} G^p(z, w)\, d\nu(z) \\
&\quad \leq A(n)(1-|w|^2)^{np} J(n, p, \alpha, w) \\
&\qquad + A(n, p, \alpha)(1-|w|^2)^{(n+1)p} \int_0^\delta r^{2(n-(n-1)p)-1}\, dr \leq A(n, p, \alpha)
\end{aligned}$$

for all $w \in B$, where Lemma 3(3) has been applied. This proves the assertion. □

5. An example

The following example shows that the result of Theorem 1 is best possible in some sense. This example is taken from [8].

Let $e_1 := (1, 0, \dots, 0) \in S$, and let

$$h(z) := P(z, e_1) = \frac{(1-|z|^2)^n}{|1-\langle z, e_1\rangle|^{2n}}. \tag{5.1}$$

Then, as shown in [8, Example 3], for each β, $0 \leq \beta \leq 1$, h^β is a nonnegative $\mathscr{M}$-superharmonic function. (In fact, it was shown that h^β is an invariant potential for all β, $0 < \beta < 1$. Notice that h itself and constant functions are $\mathscr{M}$-harmonic and hence $\mathscr{M}$-superharmonic.)

According to Lemma 3(2), for $0 < \beta \leq 1$, we have

$$\int_B (1-|z|^2)^{\alpha-1} (h^\beta)^p(z)\, d\nu(z) = J(n, \alpha, \beta p, e_1) = \infty, \tag{5.2}$$

if either $p \geq (n+\alpha)/n\beta$ or $p \leq -\alpha/n\beta$. From this we conclude that

(1) For $0 < \alpha < (n+1)/n$, we see that the upper bound $1 + \alpha/n$ of p in Theorem 1 is best possible by taking $\beta = 1$. (The case of $\alpha \geq (n+1)/n$ is not really interesting.)

(2) For $\alpha < 0$, say $\alpha = -\varepsilon$ with $\varepsilon > 0$. Then (5.2) holds for all $p \leq \varepsilon/n\beta$. Since $0 < \beta \leq 1$ is arbitrary, the number $\varepsilon/n\beta$ can be arbitrarily large. Therefore, there exists a nonnegative $\mathscr{M}$-superharmonic function that is not L^p-integrable for each $p > 0$ with respect to the measure $(1-|z|^2)^{\alpha-1}\, d\nu(z)$ in the case of $\alpha < 0$.

(3) In the case where $\alpha = 0$, the harmonic function h given in (5.1) is not L^1-integrable with respect to the measure $(1-|z|^2)^{-1}\, d\nu(z)$. Moreover, since

$\int_B (1-|z|^2)^{-1}\,d\nu(z) = \infty$, the harmonic function $h^0 \equiv 1$ is not L^p-integrable for $p > 0$ with respect to the above measure.

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REFERENCES

1 D. H. Armitage, *Further results on the global integrability of superharmonic functions*, J. London Math. Soc. (2) **6** (1972), 109–121.

2. ——, *On the global integrability of superharmonic functions in balls*, J. London Math. Soc. (2) **4** (1971), 363–373.

3. J. A. Cima and C. S. Stanton, *Admissible limits of $\mathscr{M}$-subharmonic functions*, Michigan Math. J. **32** (1985), 211–220.

4. E. B. Fabes and D. W. Stroock, *The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** (1984), 997–1016.

5. K. T. Hahn and D. Singman, *Boundary behavior of invariant Green's potentials on the unit ball in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **309** (1988), 339–354.

6. L. L. Helms, *Introduction to potential theory*, Wiley-Interscience, New York, 1969.

7. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer, New York, 1980.

8. M. Stoll, *Rate of growth of pth means of invariant potentials in the unit ball of* $\mathbb{C}^n$, J. Math. Anal. Appl. **143** (1989), 480–499.

9. D. Ullrich, *Radial limits of $\mathscr{M}$-subharmonic functions*, Trans. Amer. Math. Soc. **292** (1985), 501–518.

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