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SCRAWNY CANTOR SETS ARE NOT DEFINABLE BY TORI

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ABSTRACT. We define a Cantor set C in $\mathbf{R}^3$ to be *scrawny* if for each $p \in C$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that for each map $f: S^1 \to \operatorname{Int} B(p, \delta) - C$ there is a map $F: D^2 \to \operatorname{Int} B(p, \varepsilon)$ such that $F|\partial D^2 = f$ and $F^{-1}(C)$ is finite. We show the existence and explore some of the properties of wild scrawny Cantor sets in $\mathbf{R}^3$. We prove, among other things, that wild scrawny Cantor sets in $\mathbf{R}^3$ are not definable by solid tori.

0. INTRODUCTION: HISTORICAL BACKGROUND

This paper concerns the existence and properties of a class of wild Cantor sets in $\mathbf{R}^3$.

Any perfect, uncountable, zero-dimensional compact metric space is called a Cantor set, and any two Cantor sets are topologically equivalent. Since the publication of Antoine's Necklace in 1921 [A], it has been known that there are inequivalent embeddings of Cantor sets in $\mathbf{R}^3$; that is, there are Cantor sets C_1 and C_2 in $\mathbf{R}^3$ such that, for any homeomorphism $h: \mathbf{R}^3 \to \mathbf{R}^3$ $h(C_1) \neq h(C_2)$.

Definition. Let C be a Cantor set in $\mathbf{R}^3$. If there is a homeomorphism $h: \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(C)$ lies on a straight line then C is called *tame*. Otherwise, C is *wild*.

Definition. Let C be a Cantor set in $\mathbf{R}^3$. Let $\{\mathscr{M}_n | n \in \mathbf{N}\}$ be a sequence of finite collections $\mathscr{M}_n = \{M_{n,k} | 1 \leq k \leq m(n)\}$ of disjoint connected PL 3-manifolds-with-boundary $M_{n,k} \subseteq \mathbf{R}^3$ such that

(1) for each positive integer $\bigcup \mathscr{M}_{n+1} \subseteq \operatorname{Int} \bigcup \mathscr{M}_n$;
(2) $\bigcap \{\bigcup \mathscr{M}_n | n \in \mathbf{N}\} = C$.

Then $\{\mathscr{M}_n\}$ is called a *defining sequence* for C.

The following result is well known.

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Theorem A (Bing [Bi]). *Let C be a Cantor set in $\mathbf{R}^3$. Suppose that there is a defining sequence $\{\mathscr{M}_n | n \in \mathbf{N}\}$ for C in $\mathbf{R}^3$ such that, for each positive integer n and for each integer $1 \leq k \leq m(n)$, $M_{n,k}$ is a 3-ball. Then C is tame.*

Notation. Let $x \subseteq \mathbf{R}^3$ and let $\varepsilon > 0$. Then $B(x, \varepsilon)$ denotes the round 3-ball in $\mathbf{R}^3$ with center at x and radius ε.

Notation. Let $X \subseteq \mathbf{R}^3$ and let $\varepsilon > 0$. Then $N(X, \varepsilon)$ denotes the closure of the ε-neighborhood of X in $\mathbf{R}^3$.

Another well-known tameness theorem is

Theorem B (Bing [Bi], Homma [H]). *Let C be a Cantor set in $\mathbf{R}^3$. Suppose that for each point p of C and each $\varepsilon > 0$, there is a $\delta > 0$ such that, given any map $f: S^1 \to \operatorname{Int} B(p, \delta) - C$, there is a map $F: D^2 \to \operatorname{Int} B(p, \varepsilon) - C$ such that $F|\partial D^2 = f$. Then C is tame.*

A recent improvement on Theorem B is

Theorem C (Babich [Ba]). *Let C be a Cantor set in $\mathbf{R}^3$. Suppose that for each point p of C and each $\varepsilon > 0$, there are a positive integer n and a $\delta > 0$ such that, for each map $f: S^1 \to \operatorname{Int} B(p, \delta) - C$, there is a map $F: D^2 \to \operatorname{Int} B(p, \varepsilon)$ such that $F|\partial D^2 = f$ and $F^{-1}(C)$ consists of n points or fewer. Then C is tame.*

1. Definition and statements of main theorems

Weakening the hypothesis of Theorem C leads to the following

Definition. Let C be a Cantor set in $\mathbf{R}^3$. Suppose that for each point p of C and each $\varepsilon > 0$, there is a $\delta > 0$ such that, for each map $f: S^1 \to \operatorname{Int} B(p, \delta) - C$, there is a map $F: D^2 \to \operatorname{Int} B(p, \varepsilon)$ such that $F|\partial D^2 = f$ and $F^{-1}(C)$ is finite. Then C is *scrawny*.

In this paper we shall prove the following two theorems.

Theorem 1. *Let C be a wild scrawny Cantor set in $\mathbf{R}^3$. Then for some point x of C and some $\varepsilon > 0$, there is a nonsingular disk $D \subseteq \operatorname{Int} B(x, \varepsilon)$ such that*

(a) $D \cap C = \operatorname{Int} D \cap C = \{x\}$,
(b) $[\partial D] \neq 1$ *in* $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$.

Theorem 2. *Let C be a wild scrawny Cantor set in $\mathbf{R}^3$. There is no defining sequence $\{\mathscr{M}_n | n \in \mathbf{N}\}$ for C such that, for each positive integer n and for each integer $1 \leq k \leq m(n)$, $M_{n,k}$ is a solid torus.*

2. Example of a wild scrawny Cantor set

Before we prove theorems about wild scrawny Cantor sets in $\mathbf{R}^3$, it behooves us to show that such Cantor sets exist.

Let C be a Cantor set in $\mathbf{R}^3$ with defining sequence $\{\mathscr{M}_n\}$ such that, for each positive integer n and each $1 \leq k \leq m(n)$,

(1) $M_{n,k}$ is a solid double torus (thickened figure 8),
(2) $M_{n,k} \cap \bigcup \mathscr{M}_{n+1}$ is as shown in Figure 1.

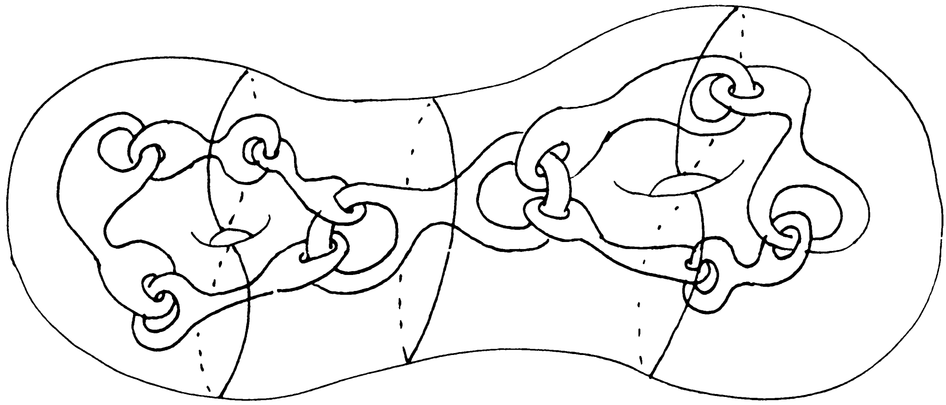

FIGURE 1

The Cantor set C (which we will call *Antoine's Eyeglasses*) is a variation on Antoine's Necklace. That C is wild can be shown by any of the many well-known proofs of the wildness of Antoine's Necklace. (See, for instance, [B1], [D], or [M].) It is evident from the construction that Antoine's Eyeglasses is scrawny.

Thus, wild scrawny Cantor sets exist in $\mathbf{R}^3$. Not all wild Cantor sets in $\mathbf{R}^3$ are scrawny. Antoine's Necklace, for example, is not scrawny. Wild scrawny Cantor sets in $\mathbf{R}^3$, then, are a nonempty proper subclass of wild Cantor sets in $\mathbf{R}^3$. Let us see what can be proved about them.

3. MAIN TOOLS

In the proofs that follow the main tools are the Loop and Sphere Theorems of Papakyriokopoulos and the following useful improvement on the Loop Theorem.

Plane Theorem (Brown and Feustel [BF]). *Let M be a noncompact 3-manifold-with-boundary, and let $f: \mathbf{R}^2 \to \operatorname{Int} M$ be a proper PL map. For each $n \in N$, let $S_n = \{x \in \mathbf{R}^2 | \, \|x\| = n\}$. Let K be a compact subset of M, and let H be a normal subgroup of $\pi_1(M - K)$. Suppose that, for any large enough positive integer n, $f(S_n) \cap K = \varnothing$ and $[f|S_n] \notin H$.*

Then there is a proper PL embedding $g: \mathbf{R}^2 \to \operatorname{Int} M$ such that, for any large enough positive integer n, $g(S_n) \cap K = \varnothing$ and $[g|S_n] \notin H$.

We shall also use the following easy consequence of Theorem A.

Theorem D (Bing [Bi]). *Let C be a wild Cantor set in $\mathbf{R}^3$. Then there are $p \in C$ and $\varepsilon > 0$ such that, for each PL 3-ball $B \subseteq \operatorname{Int} B(p, \varepsilon)$, $\partial B \cap C \neq \varnothing$.*

4. PROOFS OF MAIN THEOREMS

Theorem 1. *Let C be a wild scrawny Cantor set in $\mathbf{R}^3$. Then for some point x of C and some $\varepsilon > 0$, there is a nonsingular disk D in $\operatorname{Int} B(x, \varepsilon)$ such that*

(1) $\partial D \cap C = \varnothing$;
(2) $D \cap C = \{x\}$;
(3) $[\partial D] \neq 1$ *in* $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$.

Proof. Since C is wild, by Theorem D there is a point p of C and a $\theta > 0$ such that, for any PL 3-ball $B \subseteq \operatorname{Int} B(p, \theta)$ such that $p \in \operatorname{Int} B$, $\partial B \cap C \neq \varnothing$.

Let B be a PL 3-ball in $\operatorname{Int} B(p, \theta)$ such that $p \in \operatorname{Int} B$. Let $\eta > 0$ such that $\eta < \operatorname{dist}(\partial B, p)$ and $N(\partial B, \eta) \subseteq \operatorname{Int} B(p, \theta)$. For each $\delta > 0$, there is a PL simple closed curve α on $\partial B - C$ such that $\operatorname{diam} \alpha < \delta$ and $[\alpha] \neq 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$. For suppose, on the contrary, that there is a $\delta > 0$ such that, for each PL simple closed curve a on $\partial B - C$ such that $\operatorname{diam} \alpha < \delta$, $[\alpha] = 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$. Since $C \cap B$ is compact and zero-dimensional, there is a finite collection $\{D_i | 1 \leq i \leq n\}$ of disjoint PL disks D_i such that for each $1 \leq i \leq n$,

(a) $\partial D_i \subseteq \partial B - C$;
(b) $\operatorname{diam} D_i < \delta$;
(c) $\partial B \cap C \subseteq \bigcup\{\operatorname{Int} D_i | 1 \leq i \leq n\}$;
(d) $[\partial D_i] = 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$.

Condition (d) implies that for each $1 \leq i \leq n$, there is a PL map $f_i \colon D^2 \to \operatorname{Int} N(\partial B, \eta) - C$ such that $f_i | \partial D^2 = \partial D_i$ and $f_i^{-1}(C) = \varnothing$. Consider $\Sigma = \partial B - \bigcup\{\operatorname{Int} D_i | 1 \leq i \leq n\} \cup \bigcup\{\operatorname{Im} f_i | 1 \leq i \leq n\}$. There is a PL map $\varphi \colon S^2 \to \operatorname{Int} N(\partial B, \eta) - C$ such that $\operatorname{Im} \varphi = \Sigma$ and $[\varphi] \neq 0$ in $\pi_2(\operatorname{Int} B(p, \theta) - \{p\})$. By the Sphere Theorem of Papakyriokopoulos, there is a nonsingular PL 2-sphere S in $\operatorname{Int} N(\partial B, \eta) - C \subseteq \operatorname{Int} B(p, \theta) - C$ such that $[S] \neq 0$ in $\pi_2(\operatorname{Int} B(p, \theta) - \{p\})$. By our choices of p and θ, this is impossible.

So for each $\delta > 0$ there is a PL simple closed curve α on $\partial B - C$ such that $\operatorname{diam} \alpha < \delta$ and $[\alpha] \neq 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$. Let (α_k) be an infinite sequence of disjoint PL simple closed curves on $\partial B - C$ such that for each positive integer k,

(1) $[\alpha_k] \neq 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$,
(2) $\operatorname{diam} \alpha_k < 1/k$.

Since ∂B is compact, the set $\bigcup\{\alpha_k | k \in \mathbf{N}\}$ has at least one limit point y on ∂B. Since, for each positive integer k, $\operatorname{dist}(\alpha_k, C) < 1/k$, $y \in C$.

So there is a point y of $\partial B \cap C$ such that, for each positive integer k, there is a PL simple closed curve α_k on $\partial B - C$ such that

(a) $\operatorname{dist}(\alpha_k, y) < 1/k$,
(b) $[\alpha_k] \neq 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$.

The Cantor set C is scrawny. So there is a $\delta > 0$ such that, for each map $f \colon S^1 \to \operatorname{Int} B(y, \delta) - C$, there is a map $F \colon D^2 \to \operatorname{Int} B(y, \eta)$ such that

(1) $F | \partial D^2 = f$,
(2) $F^{-1}(C)$ is finite.

For some positive integer k, $\alpha_k \subseteq \operatorname{Int} B(y, \delta) - C$. So there is a map $F \colon D^2 \to \operatorname{Int} B(y, \eta)$ such that $F | \partial D^2 = \alpha_k$ and $F^{-1}(C)$ consists of finitely many points.

We consider the finite set $F^{-1}(C) \subseteq \operatorname{Int} D^2$. There is a point z of $F^{-1}(C)$ and a subdisk D_z of D^2 such that

(a) $F^{-1}(C) \cap D_z = \{z\}$;
(b) $z \in \operatorname{Int} D_z$ and $D_z \subseteq \operatorname{Int} D^2$;
(c) $[\partial D_z] \neq 1$ in $\pi_1(\operatorname{Int} N(\partial B, \eta) - C)$.

We may assume that $F | (D^2 - F^{-1}(C))$ is PL.

We now change F by a PL approximation outside D_z to obtain a map $F'\colon D^2 \to \operatorname{Int} B(y, \eta)$ such that

(1) $F'(D^2 - \operatorname{Int} D_z) \cap \{F(z)\} = \varnothing$;
(2) $F'|(D^2 - \{z\})$ is PL;
(3) $F'|D_z = F|D_z$.

(Note that very possible $F^{-1}(C)$ is not finite.)

Let E_k be the disk on ∂B such that $\partial E_k = \alpha_k$ and $[\partial B - E_k \cup \operatorname{Im} F'] \neq 0$ in $\pi_2(\operatorname{Int} B(p, \theta) - \{p\})$. If necessary, we adjust $\partial B - \operatorname{Int} E_k$ slightly so that $(\partial B - \operatorname{Int} E_k) \cap \{F(z)\} = \varnothing$.

Now $\partial B - \operatorname{Int} E_k \cup \operatorname{Im} F' - \{F(z)\}$ is the image of a proper PL map $g\colon \mathbf{R}^2 \to \operatorname{Int} N(\partial B, \eta) - \{F(z)\}$ such that

(a) $\lim_{\|t\| \to \infty} \{g(t) | t \in \mathbf{R}^2\} = F(z)$,
(b) for some positive integer m, $g^{-1}(C) \cap \{t \in \mathbf{R}^2 | \, \|t\| > m\} = \varnothing$.

Let M be a locally regular neighborhood of $\operatorname{Im} g$ in $\operatorname{Int} N(\partial B, h)$ such that

(1) $\operatorname{Cl}(M) = M \cup \{F(z)\}$,
(2) for some $\xi > 0$, $\operatorname{Int} B(F(z), \xi) \cap (M \cap C) = \varnothing$.

Let $K = M \cap C$. Then K is a compact subset of M. Let H be the kernel of the inclusion-induced homomorphism $\pi_1(M - K) \to \pi_1(\operatorname{Int} N(\partial B, \eta) - C)$. Then H is a normal subgroup of $\pi_1(M - K)$. For each positive integer m, let T_m be the set $\{t \in \mathbf{R}^2 | \, \|t\| \geq m\}$. Then, for m large enough, $g^{-1}(C) \cap T_m = \varnothing$ and $[g|\partial T_m] \notin H$. For each positive integer m, let $S_m = \partial T_m$.

By the Plane Theorem of Brown and Feustel, there is a proper PL embedding in $h\colon \mathbf{R}^2 \to M$ such that, for any large enough positive integer m, $[h|S_m] \notin H$. Since $h\colon \mathbf{R}^2 \to M$ is a proper embedding, $\operatorname{Cl}(\operatorname{Im} h) = \operatorname{Im} h \cup \{F(z)\}$. Thus there is an embedding $h'\colon S^2 \to M \cup \{F(z)\}$, defined by

(a) $h'|(\mathbf{R}^2 - \{\infty\}) = h$,
(b) $h'(\infty) = F(z)$.

For each positive integer m, let D_m be the nonsingular disk $h'(T_m \cup \{\infty\})$. Let $x = F(z) = h'(\infty)$. Choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \operatorname{Int} N(\partial B, \eta)$. For m large enough, $D_m \subseteq \operatorname{Int} B(x, \varepsilon)$ and $D_m \cap C = \operatorname{Int} D_m \cap C = \{x\}$. Choose m large enough and let $D = D_m$. Then D is a nonsingular disk in $\operatorname{Int} B(x, \varepsilon)$ such that

(1) $D \cap C = \operatorname{Int} D \cap C = \{x\}$,
(2) $[\partial D] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$.

This proves Theorem 1.

Theorem 2. *Let C be a wild scrawny Cantor set in $\mathbf{R}^3$. Then C is not definable by solid tori.*

Proof. Suppose, on the contrary, that C is definable by solid tori. By Theorem 1, for some point x of C and some $\varepsilon > 0$, there is a nonsingular disk D in $\operatorname{Int} B(x, \varepsilon)$ such that

(1) $D \cap C = \operatorname{Int} D \cap C = \{x\}$,
(2) $[\partial D] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$.

It follows from condition (2) above that, for any $\eta > 0$ small enough so that $\partial D \cap B(x, \eta) = \varnothing$ and for any PL 3-ball B in $\operatorname{Int} B(x, \eta)$ such that $x \in \operatorname{Int} B$, $\partial B \cap C \neq \varnothing$. Let $\eta > 0$ be small enough that $\partial D \cap B(x, \eta) = \varnothing$.

Since C is definable by solid tori, there is a collection $\{V_n | n \in \mathbf{N}\}$ of solid tori such that, for each positive integer n,

(a) $\operatorname{diam} V_n < 1/n$,
(b) there is a PL 3-ball B_n in $\operatorname{Int} V_n$ such that $V_{n+1} \subseteq \operatorname{Int} B_n$;
(c) $\partial V_n \cap C = \varnothing$;
(d) $x \in \operatorname{Int} V_n$ and $\bigcap\{V_n | n \in N\} = \{x\}$;
(e) $V_1 \subseteq \operatorname{Int} B(x, \eta)$.

Since $D \cap C = \{x\}$, for all $n \in \mathbf{N}$, $D \cap V_n \neq \varnothing$. For large enough n, $V_n \cap D \subseteq \operatorname{Int} D$. Let $n \in \mathbf{N}$ be large enough so that $V_n \cap D \subseteq \operatorname{Int} D$. We assume $D - \{x\}$ is PL and that, for each positive integer k, D is in general position with respect to ∂V_k. Thus, for all $k \geq n$, $\partial V_k \cap D$ consists of a finite number of disjoint PL simple closed curves in $\operatorname{Int} D$.

Let $k \geq n$; then $D \cap \partial V_k$ consists of finitely many PL simple closed curves in $\operatorname{Int} D$. Since $x \in D \cap V_k$, there is an innermost component σ of $D \cap \partial V_k$ such that

(1) $x \in \operatorname{Int} D_\sigma$, where D_σ is the disk in $\operatorname{Int} D$ such that $\partial D_\sigma = \sigma$;
(2) for each component γ of $\operatorname{Int} D_\sigma \cap \partial V k$, $x \notin D_\gamma$.

Since $[\partial D_\sigma] = [\partial D]$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$, $[\partial D_\sigma] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$. We assume that D_σ has been chosen so that the number of components of $\operatorname{Int} D_\sigma \cap \partial V_k$ is minimal. Hence $\operatorname{Int} D_\sigma \cap \partial V_k = \varnothing$.

For suppose that $\operatorname{Int} D_\sigma \cap \partial V_k \neq \varnothing$. Let $\{\gamma_j | 1 \leq j \leq m\}$ be the set of components of $\operatorname{Int} D_\sigma \cap \partial V_k$, and for each $1 \leq j \leq m$, let D_j be the disk in $\operatorname{Int} D_\sigma$ such that $\partial D_j = \gamma_j$. Since for each $1 \leq j \leq m$ $[\gamma_j] = 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$ and since ∂V_k is a torus and is incompressible in $\operatorname{Int} B(x, \varepsilon) - C$, for each $1 \leq j \leq m$ there is a disk E_j on ∂V_k such that $\partial E_j = \gamma_j$. For some $1 \leq j \leq m$, $E_j \cap \bigcup\{\gamma_j | i \neq j\} = \varnothing$. Consider the disk $D'_\sigma = D_\sigma - D_j \cup E_j$. D'_σ has all the virtues of D_σ, and $D'_\sigma \cap \partial V_k$ has fewer components than $D_\sigma \cap \partial V_k$. This is impossible, by our choice of D_σ.

So $D_\sigma \cap \partial V_k = \partial D_\sigma = \sigma$.

We consider $D_\sigma \cap \partial V_{k+1}$. Let $\{\theta_i | 1 \leq i \leq t\}$ be the set of those components of $D_\sigma \cap \partial V_{k+1}$ such that, for each $1 \leq i \leq t$, $[\theta_i] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$. We assume that D_σ has been chosen so that the number of components of $D_\sigma \cap \partial V_{k+1}$ is minimal. Hence $D_\sigma \cap \partial V_{k+1} = \bigcup\{\theta_i | 1 \leq i \leq t\}$. Let the numbering be chosen so that, for all $1 \leq i \leq t$, $\theta_{i+1} \subseteq \operatorname{Int} D_i$, where D_i is the disk in $\operatorname{Int} D_\sigma$ such that $\partial D_i = \theta_i$.

For each $1 \leq i \leq t$, $[\theta_i] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$. Hence, for each $1 \leq i \leq t$, $[\theta_i] \neq 1$ in $\pi_1(\partial V_{k+1})$. Since ∂V_{k+1} is a torus and since, for each $1 \leq i \neq j \leq t$, $\theta_i \cap \theta_j = \varnothing$, each θ_i is parallel to each θ_j on ∂V_{k+1}. That is, for each $1 \leq i \leq t$, $[\theta_i] = [\theta_j]$ in $\pi_1(\partial V_{k+1})$.

Since $\theta_1 \cup \sigma$ bounds an annulus in $V_k - \operatorname{Int} V_{k+1}$, $[\theta_1] = [\sigma]$ in $\pi_1(V_k - \operatorname{Int} V_{k+1})$. Since there is a PL 3-ball B_k in $\operatorname{Int} V_k$ such that $V_{k+1} \subseteq \operatorname{Int} B_k$, $[\sigma] = 1$ in $\pi_1(V_k - \operatorname{Int} V_{k+1})$. Since $[\theta_1] \neq 1$ in $\pi_1(\partial V_{k+1})$ and $[\theta_1] = 1$ in $\pi_1(V_k - \operatorname{Int} V_{k+1})$, θ_1 is a longitude of V_{k+1}.

The simple closed curve θ_t bounds the property embedded disk D_t in V_{k+1}, and $[\theta_t] \neq 1$ in $\pi_1(\partial V_{k+1})$. So θ_t is a meridian of V_{k+1}.

But we have already shown that $[\theta_t] = [\theta_1]$ in $\pi_1(\partial V_{k+1})$. This is a contradiction.

So C is not definable by solid tori. q.e.d.

5. Further observations and an example

We begin with a definition suggested by Theorem 1.

Definition. Let C be a Cantor set in $\mathbf{R}^3$ and let $x \in C$. If, for some $\varepsilon > 0$ there is a nonsingular disk D in $\operatorname{Int} B(x, \varepsilon)$ such that $D \cap C = \operatorname{Int} D \cap C = \{x\}$ and $[\partial D] \neq 1$ in $\pi_1(\operatorname{Int} B(x, \varepsilon) - C)$, then we say that C is *sliceable* at x.

Definition. Let C be a Cantor set in $\mathbf{R}^3$ and let $X = \{x \in C | C$ is sliceable at $x\}$. Then X is called the *sliceable set* of C.

Theorem D suggests the following

Definition. Let C be a Cantor set in $\mathbf{R}^3$. Let Y be the set $\{x \in C|$ for some $\varepsilon > 0$, for any PL 3-ball B in $\operatorname{Int} B(x, \varepsilon)$, $\partial B \cap C \neq \varnothing\}$. Then Y is called the *savage set* of C.

For any Cantor set C, the savage set Y is a dense subset of the wild set of C. Antoine's Eyeglasses is an example of a wild scrawny Cantor set in which $C = Y = X$. We have shown that, for any wild scrawny Cantor set C in $\mathbf{R}^3$, each point of Y is a limit point of X. Is it true that, for any wild scrawny Cantor set C in $\mathbf{R}^3$ such that $C = Y$, C is also equal to X? We give an example to show that it is false.

Let C be the Cantor set in $\mathbf{R}^3$ with defining sequence $\{\mathscr{M}_n\}$ such that, for each positive integer n and each $1 \leq k \leq m(n)$,

(1) $M_{n,k}$ is either a solid torus or a solid double torus;

(2) if $M_{n,k} \in \mathscr{M}_n$ is a solid torus, then $M_{n,k} \cap \bigcup \mathscr{M}_{n+1}$ is as shown in Figure 2;

(3) if $M_{n,k}$ is a solid double torus, then $M_{n,k} \cap \bigcup \mathscr{M}_{n+1}$ is as shown in Figure 1;

(4) $\mathscr{M}_1 = \{M_{1,1}\}$ consists of one solid torus.

There is a point p of C such that the sequence of three-manifolds-with-boundary $\{M_{n,k} | n \in \mathbf{N},\ M_{n,k} \in \mathscr{M}_n$, and $p \in M_{n,k}\}$ is a sequence of solid tori. The proof of Theorem 1 shows that C is not sliceable at p.

Thus it is not true, in general, that a wild scrawny Cantor set C in $\mathbf{R}^3$ is sliceable at each point of its savage set.

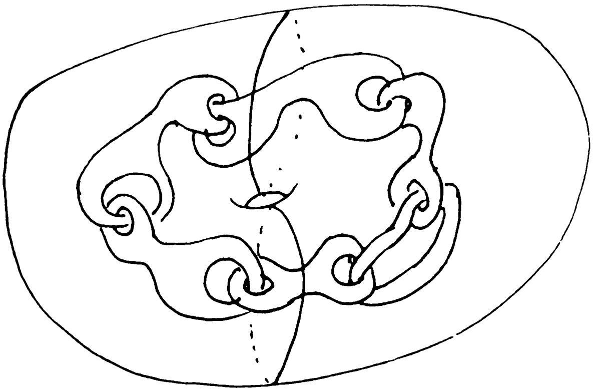

FIGURE 2

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