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UTILITY FUNCTIONS ON PARTIALLY ORDERED TOPOLOGICAL GROUPS

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ABSTRACT. Let $(X, +, \tau)$ be a locally compact abelian group endowed with a translation-invariant, strongly continuous, and separable strict partial ordering "$<$." Then, there exists a continuous numerical representation for "$<$." The proof leans on the concept of Haar measure.

1. INTRODUCTION

The numerical representation of preordered sets is a tool for decision making, where the preference of decision makers are formalized with the help of utility functions that transform preferences into numerical scales.

The problem of finding characterizations of the existence of a numerical representation on an ordered set was posed long ago by Cantor [1, 2]. Classical works on this subject are those by Eilenberg [5], Nachbin [12], and Fishburn [6]. Further studies on the general case of preordered sets appear in Mehta [9, 10] and Herden [7, 8].

In the framework of Mathematical Economics, some techniques related to measure theory have been considered in the study of utility indicators by Neuefeind [13], Mount and Reiter [11], and Chichilnisky [3, 4].

2. NUMERICAL REPRESENTATION OF A PARTIALLY ORDERED TOPOLOGICAL GROUP

Let $(X, +, \tau)$ be a locally compact abelian group with Haar measure μ, endowed with a translation-invariant strict partial ordering "$<$" (i.e., "$<$" is irreflexive and transitive). Let "0" be the null element for the group operation "+", and denote by B_0 a compact neighbourhood of 0 in (X, τ).

The order "$<$" is said to be *separable* if there exists a countable subset $D \subset X$ such that for every $a < b$, $a, b \in X$, there exists $d \in D$ with $a < d < b$. It is called *strongly continuous as regards* τ (see Peleg [14]) if for every $x \in X$ the sets $L(x) = \{a \in X ; a < x\}$ and $G(x) = \{b \in X ; x < b\}$ are open in (X, τ). A *numerical representation* (or "*utility function*," in Economics) is a continuous

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map $u\colon (X, \tau) \to \mathbb{R}$ such that for every a, $b \in X$, $a < b \Rightarrow u(a) < u(b)$. Let L be a subset of X, ∂L its boundary, $\mathrm{Int}(L)$ its interior, $\mathrm{Cl}(L)$ its closure and $\mathscr{X}_L$ the characteristic function of L in X.

Theorem. *Let $(X, +, \tau)$ be a locally compact abelian group endowed with a translation-invariant, strongly continuous, and separable strict partial ordering "$<$." Then $(X, <, \tau)$ admits a continuous numerical representation.*

Proof. Let us prove that there exists a map $A\colon X \to \mathbb{R}$ such that the function $U\colon X \to \mathbb{R}$, defined as $U(x) = \int_{x+B_0} A(\cdot)\, d\mu(\cdot)$, is a numerical representation of $(X, <, \tau)$. To construct A, first consider a subset $D = (d_n)_{n\in\mathbb{N}} \subset X$ corresponding to the separability of $<$. Consider a finite and strictly positive measure δ defined on $\mathscr{P}(\mathbb{N})$. Now, define the measure δ^* on $\mathscr{P}(D)$ by taking $\delta^*(S) = \delta(T)$, where $S = (d_t)_{t\in T}$, $S \subset D$, $T \subset \mathbb{N}$. Finally extend the measure δ^* to a measure v on $\mathscr{P}(X)$ such that $v(E) = \delta^*(E \cap D)$, $E \subset X$.

Define A as $A(x) = \sup\{(v(L(d));\ d \in D,\ d < x\}$. It is not hard to see that A is lower-semicontinuous, hence μ-measurable, and bounded. So the function $U(x) = \int_{x+B_0} A(\cdot)\, d\mu(\cdot)$ is well defined.

To see that U represents $(X, <, \tau)$ take x, $y \in X$ such that $x < y$. Then, since "$<$" is translation-invariant, it follows that $x + b < y + b$ for every $b \in B_0$, and by definition of the map A, we have $A(x + b) < A(y + b)$ $(b \in B_0)$. Therefore $U(x) < U(y)$. Now, observe that if $(x_\alpha)_{\alpha\in J}$ is a net on X converging to $x_0 \in X$, then $\mathscr{X}_{x_\alpha+B_0}$ converges to $\mathscr{X}_{x_0+B_0}$ μ-almost everywhere. To see this, take $z \in X$, $z \notin \partial(x_0 + B_0)$. Notice that $\mu(\partial(x_0 + B_0)) = 0$. The following two cases may occur: (i) $z \notin \mathrm{Cl}(x_0 + B_0)$; (ii) $z \in \mathrm{Int}(x_0 + B_0)$.

Let us see that in both cases the net $(\mathscr{X}_{x_\alpha+B_0}(z))_{\alpha\in J}$ converges to $\mathscr{X}_{x_0+B_0}(z)$. In case (i), assume that $[\mathscr{X}_{x_\alpha+B_0}(z)]_{\alpha\in J}$ does not converge to $\mathscr{X}_{x_0+B_0}(z) = 0$. Then pick up a subnet $(x_\beta)_{\beta\in K} \subset (x_\alpha)_{\alpha\in J}$ and a family of points $(b_\beta)_{\beta\in K} \subset B_0$ such that $z = x_\beta + b_\beta$ $(\beta \in K)$ $(\Rightarrow \mathscr{X}_{x_\beta+B_0}(z) = 1)$. Since B_0 is compact and $(x_\alpha)_{\alpha\in J}$ is convergent, there is no loss of generality in assuming that the net $(b_\beta)_{\beta\in K}$ converges to some point $b_0 \subset B_0$. Hence $z = x_0 + b_0 \in x_0 + B_0$. Contradiction. In case (ii) observe that $z = x_0 + b_0$ for some $b_0 \in \mathrm{Int}(B_0)$. Thus, there exists $\alpha_0 \in J$ such that for every $\alpha \geq \alpha_0$, it is $z = x_\alpha + b_\alpha \in x_0 + B_0$ (being $b_\alpha = b_0 + (x_0 - x_\alpha)$). Therefore $\mathscr{X}_{x_\alpha+B_0}(z) = 1 = \mathscr{X}_{x_0+B_0}(z)$ for every $\alpha \geq \alpha_0$. Finally, let us check the continuity of U: Consider a net $(x_\alpha)_{\alpha\in J}$ on X, convergent to $x_0 \in X$. Take a compact subset $B \subset X$ and $\alpha_0 \in J$ such that $x_\alpha + B_0 \subset B$ for every $\alpha \geq \alpha_0$. Now apply the dominated convergence theorem and the previous fact on μ-convergence to the functions $A(\cdot)\mathscr{X}_{x_\alpha+B_0}(\cdot)$ $(\alpha \geq \alpha_0)$ and conclude that the net $(U(x_\alpha))_{\alpha\in J}$ converges to $U(x_0)$. $\square$

Remark. The above result can be extended to the case of complete preorderings for which the indifference classes are μ-negligible.

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