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$\mathbb{Z}_2$-FIXED SETS OF STATIONARY POINT FREE $\mathbb{Z}_4$-ACTIONS

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ABSTRACT. In this work we consider the question: Which classes in the unoriented bordism group of free $\mathbb{Z}_2$-actions can be realized as the $\mathbb{Z}_2$-fixed set of stationary point free $\mathbb{Z}_4$-action on a closed manifold with $\mathbb{Z}_2$-fixed point set having constant codimension k ?

1. INTRODUCTION

In [3] Capobianco studied the fixed sets of involutions. He showed that the set of classes in the Thom bordism group $\mathcal{N}_n$ formed by manifolds that can be realized as the fixed set of an involution with the fixed set having codimension k is $\mathcal{N}_n$ if k is even and is the subgroup of classes in $\mathcal{N}_n$ with zero Euler characteristic if k is odd where $2 \leq k \leq n$.

A stationary point free $\mathbb{Z}_4$-action is a $\mathbb{Z}_4$-action with every isotropy subgroup being either $\mathbb{Z}_2$ or the unit subgroup. Given a closed manifold with a stationary point free $\mathbb{Z}_4$-action, one can consider the fixed point set of the action restricted to $\mathbb{Z}_2$ with the action induced by the $\mathbb{Z}_4$-action on it. So, one obtains an element in the unoriented bordism group of free $\mathbb{Z}_2$-actions, and it will be called the $\mathbb{Z}_2$-fixed point set of the $\mathbb{Z}_4$-action.

In this work we consider the question: Which classes in the unoriented bordism group of free $\mathbb{Z}_2$-actions can be realized as the $\mathbb{Z}_2$-fixed set of stationary point free $\mathbb{Z}_4$-action on a closed manifold with $\mathbb{Z}_2$-fixed point set having constant codimension k?

Denote by C_n^k the set of classes in the n-dimensional bordism group of $\mathbb{Z}_2$-free actions that can be realized as the $\mathbb{Z}_2$-fixed point set of a stationary point free $\mathbb{Z}_4$-action on a closed $(n+k)$-manifold. The main result is the following:

Theorem.
(a) $C_n^1 = (0)$;
(b) $C_n^k = \mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ *if* k *is even and* $n \geq 0$;
(c) $C_n^k = \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1]$ *if* k *odd and* $2 < k \leq n-1$, *where*

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$\mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ is the n-dimensional group of $\mathbb{Z}_2$-free actions and χ_* is the set of classes in $\mathcal{N}_*$ with zero Euler characteristic.

2. Even codimension

Let $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\})$ be the unoriented bordism group of stationary point free $\mathbb{Z}_4$-actions and $\mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})$ the unoriented bordism group of free $\mathbb{Z}_2$-actions. There is the restriction homomorphism

$$\rho: \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}) \to \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}, \mathbb{Z}_2\}),$$

which assigns to $[M, T]$ the class $[M, T^2]$. The $\mathbb{Z}_2$-fixed point set of $[M, T]$ in $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\})$ is the free involution $[F_{T^2}M, T']$, where $F_{T^2}M$ is the fixed point set of $\rho([M, T]) = [M, T^2]$ and $T' \equiv T|F_{T^2}M$.

Next, consider the $\mathcal{N}_*$-module homomorphism

$$F_{\mathbb{Z}_2}: \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}) \to \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\}),$$

which assigns to $[M, T]$ the class of the $\mathbb{Z}_2$-fixed point set of $[M, T]$. The objective of this work is to study the homomorphism $F_{\mathbb{Z}_2}$. We denote the set of classes in $\mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ that can be realized as the $\mathbb{Z}_2$-fixed set of a stationary point free $\mathbb{Z}_4$-action $[V^{n+k}, T]$ by C_n^k.

There is a $\mathbb{Z}_4$-action on a sphere $[S^{r-1+2j}, T]$, where T is given by $T(x_1, \dots, x_r, z_1, \dots, z_j) = (-x_1, \dots, -x_r, iz_1, \dots, iz_j)$ with $i = \sqrt{-1}$ whose $\mathbb{Z}_2$-fixed point set is $[S^{r-1}, -1]$. It is well known that the classes $[S^{r-1}, -1]$ form a basis for $\mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})$ as an $\mathcal{N}_*$-module; therefore, if $k = 2j$ even we have that the image of $F_{\mathbb{Z}_2}$ is $\mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ and $C_n^k = \mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ for all k even. Thus, we are reduced to the case of k being odd.

3. Codimension one

Let $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\})$ be the relative bordism group of $\mathbb{Z}_4$-actions with isotropy group $\{1\}$ or $\mathbb{Z}_2$ on manifolds with boundary for which the action is free on the boundary. There is an exact sequence

$$\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}) \to \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\}) \xrightarrow{\partial} \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$$

and an isomorphism

$$F: \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\}) \to \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty))$$

with

$$\mathcal{N}_{*-k}^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty)) \cong \mathcal{N}_{*-k}(\mathrm{BO}_k(C^\infty) \times_{\mathbb{Z}_2} E\mathbb{Z}_2),$$

where

$$\mathrm{BO}_k(C^\infty) \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \cong \mathrm{BSO}_k \times B\mathbb{Z}_4 \quad \text{for } k \text{ odd}$$

(see [1, p. 85]).

The boundary homomorphism ∂ sends the $k = 1$ summand isomorphically to $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$. This says that for $k = 1$, $C_n^1 = (0)$. Therefore, we may assume $k > 1$ and odd.

4. A construction

Being given a closed manifold with a free $\mathbb{Z}_4$-action $[N^p, T]$ and an involution $[W^q, t]$, one can form a quotient $(N^p \times W^q)/(T^2 \times t)$ with the induced $\mathbb{Z}_4$-action $T \times 1$. The $\mathbb{Z}_2$-fixed point set is $(N/T^2) \times F_t W$ with involution $T \times 1$. If W is closed, then also $(N \times W)/(T^2 \times t)$ is closed, and if W has boundary on which t is free then $T \times 1$ acts freely on the boundary $\partial((N \times W)/(T^2 \times t)) = (N \times \partial W)/(T^2 \times t)$.

Lemma 4.1. *The map*

$$\varphi : \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\}) \otimes_{\mathcal{N}_*} \mathcal{N}_*(\mathrm{BSO}_k) \to \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty)),$$

which assigns to $[N, T] \times [P, \xi]$ *the class of* $[(N \times D\xi)/(T^2 \times -1), (T \times 1)]$ *is an isomorphism for all* k *odd.*

Proof. For all k odd, we have

$$\begin{aligned}\mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty)) &\cong \mathcal{N}_*(\mathrm{BO}_k(C^\infty) \times_{\mathbb{Z}_2} E\mathbb{Z}_2)\\ &\cong \mathcal{N}_*(\mathrm{BSO}_k \times B\mathbb{Z}_4) \quad \text{(see [1, p. 86])}\\ &\cong \mathcal{N}_*(\mathrm{BSO}_k) \otimes_{\mathcal{N}_*} \mathcal{N}_*(B\mathbb{Z}_4),\end{aligned}$$

by Kunneth theorem.

Next, consider the homomorphism

$$F_c : \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\}) \to \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\}),$$

which sends $[N, T]$ to $[N/T^2, T]$. Thus we have

Theorem 4.2. *The image of the homomorphism* F_c *is the* $\mathcal{N}_*$*-submodule generated by the free involutions* $[RP(2n)][S^0, -1]$ *and* $[\mathbb{C}P(n)][S^1, -1]$, *where* n *runs through the nonnegative integers.*

Proof. First, we recall that $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$ is freely generated as an $\mathcal{N}_*$-module by extensions of the antipodal actions on even-dimensional spheres, $y_{2n} = [S^{2n} \times_{\mathbb{Z}_2} \mathbb{Z}_4, 1 \times i]$ and $y_{2n+1} = [S^{2n+1}, i]$ where $i = \sqrt{-1}$.

Now, calculating the image of F_c on the generators of $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$, we have

$$\begin{aligned}F_c([S^{2n} \times_{\mathbb{Z}_2} \mathbb{Z}_4, 1 \times i]) &= [(S^{2n} \times_{\mathbb{Z}_2} \mathbb{Z}_4)/(1 \times -1), 1 \times i]\\ &= [RP(2n)][S^0, -1]\end{aligned}$$

and

$$F_c([S^{2n+1}, i]) = [S^{2n+1}/-1, i] = [RP(2n+1), i].$$

Next, to see that $[RP(2n+1), i] = [\mathbb{C}P(n)][S^1, -1]$, consider f: $RP(2n+1) \to B\mathbb{Z}_2$ classifying the $\mathbb{Z}_2$-bundle $S^{2n+1} \to RP(2n+1)$ and g: $RP(2n+1)/\mathbb{Z}_2 \to B\mathbb{Z}_4$ classifying the $\mathbb{Z}_4$-bundle $S^{2n+1} \to RP(2n+1)/\mathbb{Z}_2$, where $\mathbb{Z}_4$ acts on S^{2n+1} by multiplication by $i = \sqrt{-1}$. Let p: $RP(2n+1) \to RP(2n+1)/\mathbb{Z}_2$ be the canonical projection. Thus, we have the commutative diagrams

$$\begin{array}{ccc} RP(2n+1) & \xrightarrow{f} & B\mathbb{Z}_2 \\ {\scriptstyle p}\downarrow & & \downarrow \\ RP(2n+1)/\mathbb{Z}_2 & \xrightarrow{g} & B\mathbb{Z}_4 \end{array}$$

and

$$\begin{array}{ccc} H^*(RP(2n+1); \mathbb{Z}_2) & \xleftarrow{f^*} & H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \\ {\scriptstyle p^*}\uparrow & & \uparrow \\ H^*(RP(2n+1)/\mathbb{Z}_2; \mathbb{Z}_2) & \xleftarrow{g^*} & H^*(B\mathbb{Z}_4; \mathbb{Z}_2) \end{array}$$

Therefore, since f^* and g^* are isomorphisms in dimensions $\leq 2n+1$, we have $H^*(RP(2n+1)/\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, x_2]/(x_1^2 = 0, x_2^{n+1} = 0)$, where $x_1 = g^*(\alpha)$ and $x_2 = g^*(\beta)$ being α, β the generators of $H^*(B\mathbb{Z}_4; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta]/(\alpha^2 = 0, \beta^{n+1} = 0)$.

Now, considering the map

$$RP(2n+1)/\mathbb{Z}_2 \xrightarrow{g} B\mathbb{Z}_4 \to B\mathbb{Z}_2$$

classifying the involution $[RP(2n+1), i]$, we see that the characteristic class of this involution is $c = x_1$ and $c^j = 0$ for all $j > 1$.

Next, let ξ be the linear bundle over complex projective $2n$-space $\mathbb{C}P(n)$. Thus, S^{2n+1} can be identified with the total space of the sphere bundle of ξ, i.e., $S^{2n+1} \equiv S(\xi)$. In the same way, we have $RP(2n+1) \equiv S(\xi \otimes \xi)$ and $RP(2n+1)/\mathbb{Z}_2 \equiv S(\xi \otimes \xi \otimes \xi \otimes \xi)$. The tangent bundle of $S(\xi \otimes \xi \otimes \xi \otimes \xi)$ is equivalent to $\pi^*(\tau(\mathbb{C}P(n))) \oplus \pi^*(\xi \otimes \xi \otimes \xi \otimes \xi)$, where $\pi\colon RP(2n+1)/\mathbb{Z}_2 \to \mathbb{C}P(n)$ is the projection. Thus, the Stiefel-Whitney class of $RP(2n+1)/\mathbb{Z}_2$ is

$$\begin{aligned} w(RP(2n+1)/\mathbb{Z}_2) &= \pi^*(w(\mathbb{C}P(n)))\pi^*(w(\xi \otimes \xi \otimes \xi \otimes \xi)) \\ &= (1+x_2)^{n+1}(1+4x_2) = (1+x_2)^{n+1}. \end{aligned}$$

On the other hand, considering the involution $[\mathbb{C}P(n)][S^1, -1]$, the characteristic class of this involution is given by $c' = 1 \times \alpha_1$, where α_1 is the generator of $H^1(RP(1); \mathbb{Z}_2)$ and the Stiefel-Whitney class of $\mathbb{C}P(n) \times RP(1)$ is

$$w(\mathbb{C}P(n) \times RP(1)) = \sum_{i=0}^{n} \binom{n+1}{i} \alpha_2^i \times 1,$$

where α_2 is the generator of $H^2(\mathbb{C}P(n); \mathbb{Z}_2)$.

Therefore, it is easy to see that all of the involutions numbers of the two involutions are the same. Hence the theorem follows.

5. An upper bound

Consider the $\mathcal{N}_*$-module homomorphism

$$\overline{F}_{\mathbb{Z}_2}\colon \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\}) \to \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})$$

mapping the class of $[M, T]$ into the class of $[M, T^2]$ and recall that

$$\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\}) \cong \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty)).$$

Now, considering

$$\bigoplus_{\substack{k=1,\\ k \text{ odd}}}^{*} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty)),$$

we have

Theorem 5.1.

$$\overline{F}_{\mathbb{Z}_2}\left(\bigoplus_{\substack{k=1\\ k \text{ odd}}}^{*} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(\{\{1\}\})(\mathrm{BO}_k(C^\infty))\right) = \mathcal{N}_*[S^0, -1] + \mathcal{N}_{*-1}[S^1, -1].$$

Proof. Using the isomorphism of the Lemma 4.1, we may calculate the image of $\overline{F}_{\mathbb{Z}_2}$ on the elements $[D\xi^k \times_{\mathbb{Z}_2} N, 1 \times t]$, where $[N, t]$ runs through a set of generators of $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$. Then we have that

$$\begin{aligned}\overline{F}_{\mathbb{Z}_2}([D\xi^k \times_{\mathbb{Z}_2} (S^{2n} \times_{\mathbb{Z}_2} \mathbb{Z}_4), 1 \times 1 \times i]) &= [P][(S^{2n} \times_{\mathbb{Z}_2} \mathbb{Z}_4)/(1 \times -1), 1 \times i] \\ &= [P][RP(2n)][S^0, -1],\end{aligned}$$

where P is the base space of ξ^k, and

$$\begin{aligned}\overline{F}_{\mathbb{Z}_2}([D\xi^k \times_{\mathbb{Z}_2} S^{2n+1}, 1 \times i]) &= [P][RP(2n+1), i] \\ &= [P][\mathbb{C}P(n)][S^1, -1],\end{aligned}$$

as in the proof of the Theorem 4.2. Thus, it follows that $\mathcal{N}_*[S^0, -1] + \mathcal{N}_{*-1}[S^1, -1]$ contains the image.

Now, taking the free $\mathbb{Z}_4$-actions $[S^0 \times_{\mathbb{Z}_2} \mathbb{Z}_4, 1 \times i]$, $[S^1, i]$, and the bundle $[P, \xi^k]$ with the base space P consisting of a single point, we see that

$$\overline{F}_{\mathbb{Z}_2}([D\xi^k \times_{\mathbb{Z}_2} (S^0 \times_{\mathbb{Z}_2} \mathbb{Z}_4), 1 \times 1 \times i]) = [S^0, -1]$$

and

$$\overline{F}_{\mathbb{Z}_2}([D\xi^k \times_{\mathbb{Z}_2} S^1, 1 \times i]) = [S^1, -1].$$

Therefore, we have the result.

Note. By the above theorem, we see that

$$C_n^k \subset \mathcal{N}_n[S^0, -1] + \mathcal{N}_{n-1}[S^1, -1]$$

for all n, k and $k > 1$ odd.

Theorem 5.2. $\chi_n[S^0, -1] + \chi_{n-1}[S^1, -1] \subset C_n^k \subset \mathcal{N}_n[S^0, -1] + \mathcal{N}_{n-1}[S^1, -1]$ *for all* $2 < k \leq n-1$ *and* k *odd, where* χ_* *is the set of classes in* $\mathcal{N}_*$ *with zero Euler characteristic.*

Proof. Let M^n and N^{n-1} be in χ_n and χ_{n-1}, respectively. By Capobianco [3], there are involutions $[V_1^{n+k}, T_1]$ and $[V_2^{n-1+k}, T_2]$ such that the fixed point sets are M^n and N^{n-1}, respectively, for all $2 < k \leq n-1$ and k odd. Thus, the stationary point free $\mathbb{Z}_4$-action

$$[(V_1 \times \mathbb{Z}_4)/(T_1 \times -1), 1 \times i] + [(V_2 \times S^1)/(T_2 \times -1), 1 \times i]$$

has $\mathbb{Z}_2$-fixed point set $[M^n][S^0, -1] + [N^{n-1}][S^1, -1]$.

6. Euler characteristics

Let $g_*\colon \mathcal{N}_*(\mathrm{BO}_{k-1}) \to \mathcal{N}_*(\mathrm{BSO}_k)$ be the map given by $g_*([M, \xi^{k-1}]) = [M, \xi^{k-1} \oplus \det \xi^{k-1}]$; this map is well defined. In fact, calculating the Stiefel-Whitney class of the bundle $\xi^{k-1} \oplus \det \xi^{k-1}$, we have

$$\begin{aligned}w(\xi^{k-1} \oplus \det \xi^{k-1}) &= w(\xi^{k-1}) w(\det \xi^{k-1}) \\ &= (1 + w_1 + \cdots + w_{k-1})(1 + w_1) \\ &= 1 + (w_1 + w_1) + (w_2 + w_1^2) + \cdots + w_{k-1}w_1 \\ &= 1 + (w_2 + w_1^2) + \cdots + w_{k-1}w_1,\end{aligned}$$

where w_i are the Stiefel-Whitney classes of the bundle ξ^{k-1}. Thus, the first Stiefel-Whitney class of the bundle $\xi^{k-1} \oplus \det \xi^{k-1}$ is zero and $\xi^{k-1} \oplus \det \xi^{k-1}$ is orientable, i.e., the given class is in $\mathcal{N}_*(\mathrm{BSO}_k)$.

Now, recall that $H^*(\mathrm{BSO}_k, \mathbb{Z}_2) = \mathbb{Z}_2[v_2, \dots, v_k]$ and $H^*(\mathrm{BO}_{k-1}; \mathbb{Z}_2) = \mathbb{Z}_2[v'_1, v'_2, \dots, v'_{k-1}]$, where $v = 1 + v_2 + \cdots + v_k$ and $v' = 1 + v'_1 + \cdots + v'_{k-1}$ are the total universal Whitney classes in $H^*(\mathrm{BSO}_k; \mathbb{Z}_2)$ and $H^*(\mathrm{BO}_{k-1}; \mathbb{Z}_2)$, respectively. Next, let $g^*: H^*(\mathrm{BSO}_k; \mathbb{Z}_2) \to H^*(\mathrm{BO}_{k-1}; \mathbb{Z}_2)$ be the induced map given by $g^*(v) = v'(1 + v'_1)$. Since g^* is monic, [4, 17.3] implies that g^* is epic.

Now, taking $J = (j(1), j(2), \dots, j(k-1))$ a $(k-1)$-tuple of nonnegative integers with $j(1) \geq j(2) \geq \cdots \geq j(k-1)$ and considering ξ^J the bundle

$$p_1^*(\xi_{j(1)}) \oplus p_2^*(\xi_{j(2)}) \oplus \cdots \oplus p_{k-1}^*(\xi_{j(k-1)}) \oplus (p_1^*(\xi_{j(1)}) \otimes p_2^*(\xi_{j(2)}) \otimes \cdots \otimes p_{k-1}^*(\xi_{j(k-1)}))$$

over $RP^J = RP(j(1)) \times RP(j(2)) \times \cdots \times RP(j(k-1))$, where $\xi_{j(i)}$ is the canonical line bundle over the projective space $RP(j(i))$ and $p_i: RP^J \to RP(j(i))$ is the projection onto the ith factor, we have

Lemma 6.1. *The bundles ξ^J, $J = (j(1), j(2), \dots, j(k-1))$ with $j(1) \geq j(2) \geq \cdots \geq j(k-1) \geq 0$ constitute a set of generators for $\mathcal{N}_*(\mathrm{BSO}_k)$.*

Proof. The result follows by the above remarks and [5, 3.4.2].

Theorem 6.2. *The kernel of the homomorphism*

$$G: \bigoplus_{s=0}^{n} \mathcal{N}_s^{\mathbb{Z}_4}(\{\{1\}\}) \otimes_{\mathcal{N}_*} \mathcal{N}_{n-s}(\mathrm{BSO}_k) \xrightarrow{\partial\circ\varphi} \mathcal{N}_{n+k-1}^{\mathbb{Z}_4}(\{\{1\}\}) \xrightarrow{\rho} \mathcal{N}_{n+k-1}^{\mathbb{Z}_2}(\{\{1\}\})$$

is contained in the set of classes $[\alpha]$ such that $\overline{F}_{\mathbb{Z}_2}([\alpha])$ belongs to $\chi_n[S^0, -1] + \chi_{n-1}[S^1, -1]$, for all n, k odd and $k > 1$.

Proof. It is sufficient to verify the result for all the generators of $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$ and $\mathcal{N}_*(\mathrm{BSO}_k)$. Considering the even $2l$-dimensional generators of $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$ and being given the bundle $[P, \xi]$ an element in $\mathcal{N}_{n-2l}(\mathrm{BSO}_k)$, we have $G([S^{2l} \times_{\mathbb{Z}_2} \mathbb{Z}_4, 1 \times i], [P, \xi]) = [S^{2l} \times \mathbb{Z}_2, 1 \times -1][S(\xi), -1] = 0$ since $[S^{2l} \times \mathbb{Z}_2, 1 \times -1]$ is boundary in $\mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\})$; and $\overline{F}_{\mathbb{Z}_2}([S^{2l} \times_{\mathbb{Z}_2} \mathbb{Z}_4, 1 \times i], [P, \xi]) = (RP(2l) \times P)[S^0, -1]$ with $\chi(RP(2l) \times P) \equiv 0$ since the dimension of $RP(2l) \times P$ is $2l + (n - 2l) = n$ odd.

Now, considering $[RP^J, \xi^J]$ a generator of $\mathcal{N}_{n-2l-1}(\mathrm{BSO}_k)$ and odd $(2l+1)$-dimensional generators of $\mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}\})$, we have

$$G([S^{2l+1}, i], [RP^J, \xi^J]) = [S^{2l+1}, -1][S(\xi^J), -1],$$

and taking the isomorphism $\overline{F}: \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\}) \xrightarrow{\cong} \mathcal{N}_*(RP^\infty)$, we see that

$$\overline{F}([S^{2l+1}, -1][S(\xi^J), -1]) = [RP(2l+1) \to RP^\infty][RP(\xi^J) \to RP^\infty] [f: RP(2l+1) \times RP(\xi^J) \to RP^\infty],$$

by [7], where the map f classifies the bundle $[RP(2l+1) \times RP(\xi^J), \gamma^1 \otimes \gamma^2]$ with γ^1 the line bundle over $RP(2l+1)$ and γ^2 the line bundle over $RP(\xi^J)$. Calculating the Whitney number $\langle cw_{n+k-2}, \sigma_{n+k-1}\rangle$ of the map f, where $c = \alpha_{2l+1} \times 1$ and α_{2l+1} is the generator of $H^1(RP(2l+1); \mathbb{Z}_2)$, we have

$$\begin{aligned}\langle cw_{n+k-2}, \sigma_{n+k-1}\rangle &= \langle(\alpha_{2l+1} \times 1)w_{n+k-2}(RP(2l+1) \times RP(\xi^J)), \sigma_{n+k-1}\rangle \\ &= \left\langle (\alpha_{2l+1} \times 1)\left(\binom{2l+2}{2l}\alpha_{2l+1}^{2l}\right) \times \chi(RP(\xi^J)), \sigma_{n+k-1}\right\rangle.\end{aligned}$$

On the other hand,

$$\overline{F}_{\mathbb{Z}_2}([S^{2l+1}, i], [RP^J, \xi^J]) = (\mathbb{C}P(l) \times RP^J) \cdot [S^1, -1],$$

with

$$\chi(\mathbb{C}P(l) \times RP^J) = \binom{l+1}{l} \beta^l \times \chi(RP^J),$$

where β is the generator of $H^2(\mathbb{C}P(l); \mathbb{Z}_2)$. Therefore, we conclude that $\chi(\mathbb{C}P(l) \times RP^J) \equiv \langle c w_{n+k-2}, \sigma_{n+k-1} \rangle$. Thus, if $\chi(\mathbb{C}P(l) \times RP^J) \not\equiv 0$, we see that $([S^{2l+1}, i], [RP^J, \xi^J])$ is not in the kernel of G.

Theorem 6.3.

(a) $C_n^1 = (0)$;
(b) $C_n^k = \mathcal{N}_n^{\mathbb{Z}_2}(\{\{1\}\})$ *for all* $n \geq 0$ *and* k *even*;
(c) $C_n^k = \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1]$ *for* $2 < k \leq n-1$ *and* k *odd.*

Proof. Considering k odd and n even, let $[M, t] = A[S^0, -1] + B[S^1, -1]$ be in $C_n^k \subset \mathcal{N}_n[S^0, -1] + \mathcal{N}_{n-1}[S^1, -1]$ and $[V^{n+k}, T]$ a stationary point free $\mathbb{Z}_4$-action such that $[M^n, t]$ is the $\mathbb{Z}_2$-fixed point set. Then, $n+k$ is odd, so $\chi(\mathrm{V}) \equiv 0$. Since the $\mathbb{Z}_4$-action T free on $V - M$, we have that $\chi(M) \equiv 0 \bmod 4$. Then, $M/t = A + (B \times RP(1))$ in $\mathcal{N}_n$, $\chi(M/t) \equiv 0 \bmod 2$, and $\chi(B \times RP(1)) \equiv 0$, since the dimension of $B \times RP(1)$ is $n-1$ odd, imply that $\chi(A) \equiv 0$, i.e., A belongs to χ_n. One has $\chi_{n-1} = \mathcal{N}_{n-1}$ since $n-1$ is odd; therefore, we have that $C_n^k = \chi_n[s^0, -1] + \chi_{n-1}[S^1, -1]$ for k odd and n even.

Next, for k odd and n odd, we have the exact sequence and commutative diagram

$$\begin{array}{ccccc} \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}) & \longrightarrow & \mathcal{N}_*^{\mathbb{Z}_4}(\{\{1\}, \mathbb{Z}_2\}, \{\{1\}\}) & \xrightarrow{\partial} & \mathcal{N}_{*-1}^{\mathbb{Z}_4}(\{\{1\}\}) \\ & {\scriptstyle F_{\mathbb{Z}_2}} \searrow & & \swarrow {\scriptstyle \overline{F}_{\mathbb{Z}_2}} & \\ & & \mathcal{N}_*^{\mathbb{Z}_2}(\{\{1\}\}) & & \end{array}$$

Thus,

$$\begin{aligned} \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1] &\subset C_n^k \\ &\subset (\mathcal{N}_n[S^0, -1] + \mathcal{N}_{n-1}[S^1, -1]) \cap \overline{F}_{\mathbb{Z}_2}(\ker \partial) \end{aligned}$$

by the exactness of the sequence and Theorem 5.2. Further, since $\overline{F}_{\mathbb{Z}_2}(\ker \partial) \subset \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1]$ by the Theorem 6.2, we conclude that

$$\chi_n[S^0, -1] + \chi_{n-1}[S^1, -1] \subset C_n^k \subset \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1];$$

that is, $C_n^k = \chi_n[S^0, -1] + \chi_{n-1}[S^1, -1]$ for k odd and n odd.

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