PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 3, July 1992

AHLFORS FUNCTIONS ON DENJOY DOMAINS

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. Let Δ be the open unit disc. We give a characterization of a set that is the complement in Δ of the image of the Ahlfors function for some maximal Denjoy domain and ∞. As a corollary, we show by an example that there exists such a set with positive logarithmic capacity.

1. INTRODUCTION

Let Ω be a region on the Riemann sphere $\widehat{\mathbb{C}}$ that supports nonconstant bounded analytic functions and let $p \in \Omega$. Set $\mathbf{B} = \{f | f$ is holomorphic in Ω and $f(\Omega) \subset \Delta\}$ where Δ is the unit disc $\{z||z| < 1\}$. The Ahlfors function for Ω and p is the unique function F [1] in B such that

$$F'(p) = \max_{f \in \mathbf{B}} \operatorname{Re} f'(p).$$

By linear transformation, we see easily that $F(p) = 0$. It is well known [6; 1, Theorem 3] that the image $F(\Omega)$ of any Ahlfors function F covers the unit disc with the exception of a set of analytic capacity zero. We are interested in a characterization of this exceptional set that is called the *omitted set* of F. In this respect we restrict ourselves to study only Ahlfors functions whose domains of definition are maximal regions for bounded analytic functions in the sense of Rudin [10], since otherwise the above problem becomes less interesting [7]. Some examples of omitted sets of Ahlfors functions were given by several authors. Roding [9] gave an example of an omitted set consisting of two points. Minda [7] extended this example to fairly general discrete sets, and the author [13] gave examples of fairly general sets of logarithmic capacity zero.

As a first step toward the characterization of the omitted set we study the simplest case where the domain Ω is a Denjoy domain defined as follows: A planar domain $D \ni \infty$ is called a *Denjoy domain* if ∂D is a compact subset of the real axis $\mathbb{R}$. The beautiful idea of using a Denjoy domain to study the omitted set of Ahlfors function is due to Minda [7]. The main result of our paper gives a necessary and sufficient condition for a subset of the unit disc to be the omitted set of the Ahlfors function F for some maximal Denjoy domain and

Received by the editors October 24, 1990.
1991 *Mathematics Subject Classification*. Primary 30D50; Secondary 30C85.
Key words and phrases. Ahlfors function, analytic capacity, logarithmic capacity, Denjoy domain.

∞ such that F is a covering onto its image. As a corollary we give examples of omitted sets of Ahlfors functions that have positive logarithmic capacity. In [2] Fisher raised a question whether the composite function $F \circ \phi$ is an inner function on Δ where F is the Ahlfors function for a maximal domain D and $\phi: \Delta \to D$ is its uniformizer. Our examples immediately give another proof of a negative answer to this question, which was first shown by Gamelin [3, p. 93].

2. Some lemmas

Let Ω be a Denjoy domain with boundary $E \subset \mathbb{R}$ that supports nonconstant bounded analytic functions. The following well-known formula [8] gives a useful integral representation of the Ahlfors function.

Lemma 1 (Pommerenke). *The Ahlfors function F for a Denjoy domain Ω and ∞ is given by*

$$F(z) = \tanh\left(\frac{1}{4}\int_E \frac{d\zeta}{z-\zeta}\right), \qquad z \in \Omega.$$

From Lemma 1, we obtain another representation of F by using harmonic measure.

Lemma 2. *The Ahlfors function F for a Denjoy domain Ω and ∞ is given by*

$$F(z) = -i\tan\left(\frac{\pi}{4}[\omega_E(z) + i\omega_E^*(z)]\right), \qquad z \in \Omega,$$

where $\omega_E(z) = \frac{1}{\pi}\int_E \operatorname{Im}(\zeta - z)^{-1}\,d\zeta$, $z \in \Omega$, is the harmonic measure of E relative to H, the upper half plane, and ω_E^* its harmonic conjugate function.

Proof. Recall that the Poisson integral formula for a bounded harmonic function f on H is given by

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \tilde{f}(\zeta)\operatorname{Im}\frac{1}{\zeta - z}\,d\zeta$$

where $\tilde{f}$ is the nontangential limit of f. Thus,

$$\begin{aligned}\frac{1}{4}\int_E \frac{d\zeta}{z-\zeta} &= -\frac{1}{4}\int_E \left(\operatorname{Re}\frac{1}{\zeta - z} + i\operatorname{Im}\frac{1}{\zeta - z}\right)d\zeta \\ &= -\frac{\pi i}{4}(\omega_E(z) + i\omega_E^*(z)).\end{aligned}$$

In view of Lemma 1, this gives the lemma. □

Lemma 3. *Let F be the Ahlfors function for a Denjoy domain Ω and ∞. Then F satisfies the identity $F(\bar{z}) = \overline{F(z)}$ for all $z \in \Omega$. Moreover,* $\operatorname{Im} F(z) < 0$ *if and only if $z \in \Omega \cap H$.*

Proof. From Lemma 1 it is clear that the above identity holds. By Lemma 2, we have

$$\operatorname{Im} F(z) = -\operatorname{Re}\tan\left(\frac{\pi}{4}(\omega_E + i\omega_E^*)\right).$$

A calculation shows that

$$\operatorname{Im} F(z) = -\tan\frac{\pi}{4}\omega_E / \left(\cosh^2\frac{\pi}{4}\omega_E^* + \sinh^2\frac{\pi}{4}\omega_E^*\tan^2\frac{\pi}{4}\omega_E\right).$$

Since F is nonconstant, the linear measure of the boundary E is positive. Thus the harmonic measure ω_E satisfies the inequality $0 < \omega_E < 1$ on H. This trivially implies the second statement of the lemma. □

3. Denjoy domain of type (K, π)

Given a set $X \subset \mathbb{C}$ let us denote by $\mathrm{Cl}(X)$ its closure and by $\overline{X}$ the reflection of the set X in the real axis. Let Σ be the family of relatively closed subsets of Δ satisfying:

(1) K has analytic capacity zero,
(2) $K = \overline{K}$,
(3) $K \cap \mathbb{R} = \mathrm{Cl}(K \backslash \mathbb{R}) \cap \mathbb{R}$,
(4) $0 \notin K$.

The following construction of a maximal Denjoy domain is a slight extension of the one introduced by Minda [7]. For any $K \in \Sigma$ let $\pi\colon H \to \Delta_- \backslash K$ be a holomorphic universal covering with the cover transformation group Γ where $\Delta_- = \Delta \cap \overline{H}$ denotes the lower half disc. Let A be the set of points p in ∂H such that the covering π has a continuous extension to some neighborhood $(\subset \partial H)$ of p where π takes real boundary values everywhere. Since conditions (1) and (3) imply that $\Delta_- \backslash K$ has a free boundary arc, we see that Γ is a Fuchsian group of the second kind acting on H and that A is a nonempty open subset of ∂H. Replacing π with $\pi \circ \gamma$ for some $\gamma \in \mathrm{Möb}(H)$, the Möbius transformation group acting on H, if necessary, we assume here and hereafter that $\infty \in A$ and that $\pi(\infty) = 0$. This is possible by condition (4). By Schwarz reflection principle the covering π is continued holomorphically to the Denjoy domain $\Omega = H \cup A \cup \overline{H}$. We use the same notation π to denote the extended map. Since by condition (3) each component of the set $(-1, 1) \backslash K$ is a free boundary arc of $\Delta_- \backslash K$, we have an important observation that A is a Γ-invariant subset of ∂H and $\pi(A) = (-1, 1) \backslash K$. One verifies easily that the map $\pi\colon \Omega \to \Delta \backslash K$ is a holomorphic covering of $\Delta \backslash K$ with the cover transformation group Γ. Note that since π maps H to $\overline{H}$ and $\pi(\infty) = 0$, we have $\pi'(\infty) = \lim_{z \to \infty} z(\pi(z) - \pi(\infty)) > 0$. The domain Ω constructed above is called the *Denjoy domain of type* (K, π). We remark that the covering π is a nonconstant bounded analytic function on Ω.

Wee need four lemmas.

Lemma 4. *Let Ω be the Denjoy domain of type (K, π). Then Ω is simply connected if and only if $K = \varnothing$.*

Proof. We assume that Ω is simply connected, since if $K = \varnothing$ the lemma is clear. Then Ω is of the form $\widehat{\mathbb{C}} \backslash [a, b]$ for some a and $b \in \mathbb{R}$ $(a < b)$. since the interval $[a, b]$ is Γ-invariant, it is easy to see that each $\gamma \in \Gamma$ fixes a and b where Γ is the cover transformation group of the covering π. It follows from discontinuity of Γ that the group Γ is either trivial or hyperbolic cyclic. Since $\Delta \backslash K$ is conformally equivalent to the quotient surface Ω/Γ, we have a bounded univalent function $f\colon \Delta \backslash K \to S$ where S is either Δ or an annulus. Since K has analytic capacity zero, K is a removable set for bounded analytic functions [4, p. 10]. Hence f is extended to a univalent function on Δ whose image is the same S. This implies that $K = \varnothing$.

Lemma 5. *The Denjoy domain of type (K, π) is maximal.*

Proof. By definition the covering π satisfies the identity $\pi(\overline{z}) = \overline{\pi(z)}$, $z \in \Omega$. Assume that π has an analytic extension to some neighborhood U of a point $p \in \mathbb{R}$. Then the above identity implies that $\pi(U \cap \mathbb{R}) \subset \mathbb{R}$. By definition of the set $\Omega \cap \mathbb{R}$, we have $p \in \Omega \cap \mathbb{R} \subset \Omega$. Since π is a bounded analytic function on Ω, this implies that any point of $\partial\Omega$ is an essential boundary point. □

Lemma 6. *Let Ω be a maximal Denjoy domain that supports nonconstant bounded analytic functions, and let F be the Ahlfors function for Ω and ∞ with the omitted set K. If F is a covering onto its image, then $K \in \Sigma$ and Ω is the Denjoy domain of type (K, F).*

Proof. By Lemma 3, the restriction $F|_H$ is a universal covering onto the domain $\Delta_-\backslash K$. Set $K' = \mathrm{Cl}(K\backslash\mathbb{R}) \cap \Delta$ $(\subset K)$. We claim that $K' \in \Sigma$. By definition, conditions (2) and (3) defining the family Σ are trivially satisfied, and (4) is clear since $P(\infty) = 0$. Since the image of an Ahlfors function is the unit disc with the exception of a set with analytic capacity zero [1], condition (1) also holds. Hence the claim is proved.

Because the restriction $F|_H$ is a universal covering of $\Delta_-\backslash K'$, let D be the Denjoy domain of type $(K', F|_H)$. To conclude the proof, we must show that $D = \Omega$ and $K' = K$. Since $F(\Omega \cap \mathbb{R}) \subset (-1, 1)$ by Lemma 3, recalling the definition of the set $D \cap \mathbb{R}$, we obtain $\Omega \subset D$. Assume that there exists a point $p \in D\backslash\Omega$. Since the domain Ω is maximal, p is an essential boundary point of Ω. Then again the definition of the set $D \cap \mathbb{R}$ shows that F has a holomorphic extension to some neighborhood of p and that $F(p) \in (-1, 1)$. This, however, contradicts the fact that if p is an essential boundary point of Ω, then by [2, Corollary] $\limsup_{x\to p} |F(x)| = 1$. Hence $D = \Omega$ and we have $F(\Omega) = \Delta\backslash K'$, so $K' = K$ as desired. □

Lemma 7. *Let Ω be the Denjoy domain of type (K, π), and let f be the Ahlfors function for Ω and ∞. If the image $f(\Omega)$ is a proper subset of Δ, then we have $f = \pi$ and $f(\Omega) = \Delta\backslash K$.*

Proof. Let Γ be the cover transformation group of the covering π. From Lemma 1, using a fact that $\partial\Omega$ is Γ-invariant, a direct calculation shows that for $\gamma \in \Gamma$, $f \circ \gamma = \phi(\gamma) \circ f$, where $\phi(\gamma)$ is a hyperbolic element with fixed points at ± 1 of the form $\tau^{-1} \circ A_\gamma \circ \tau$, $A_\gamma(z) = e^{M(\gamma)}z$, $M(\gamma) \in \mathbb{R}$, and $\tau(z) = (1+z)/(1-z)$. Indeed, from Lemma 1 we see by calculation that the map $M: \Gamma \to \mathbb{R}$ is a group homomorphism, that is, $M(\gamma \circ \delta) = M(\gamma) + M(\delta)$ for any γ and $\delta \in \Gamma$.

First, we claim that the subgroup $M(\Gamma) \subset \mathbb{R}$ is discrete. Note that, since $\phi(\gamma)$ is a conformal automorphism of the image $f(\Omega)$, $\phi(\gamma)$ is also a bijective self-map of $\Delta\backslash f(\Omega)$ $(\neq \varnothing)$. If $M(\Gamma)$ is not discrete, then it is elementary to see that $M(\Gamma)$ is dense in $\mathbb{R}$. Considering the points $\phi(\gamma)(p)$ for all $\gamma \in \Gamma$ with some fixed point $p \in \Delta\backslash f(\Omega)$, we find that the set $\Delta\backslash f(\Omega)$ contains a dense subset of a circular arc with its end points at 1 and -1. Since $\Delta\backslash f(\Omega)$ is closed, $\Delta\backslash f(\Omega)$ contains a circular arc. This contradicts the fact that the analytic capacity of the set $\Delta\backslash f(\Omega)$ is zero. Thus, $M(\Gamma)$ is discrete and the claim is proved.

Again it is elementary to se that if $M(\Gamma)$ is discrete then $M(\Gamma)$ is cyclic. Hence the subgroup $\phi(\Gamma) \subset \mathrm{Möb}(\Delta)$ is a cyclic group $\langle\alpha\rangle$ with generator $\alpha \in \mathrm{Möb}(\Delta)$. We claim that f is Γ-invariant. If α is the identity, then $\phi(\Gamma)$ is a trivial group and we have nothing to prove. Thus we may assume that α is hyperbolic and that the quotient surface $\Delta/\langle\alpha\rangle$ is an annulus. Let $p: \Delta \to \Delta/\langle\alpha\rangle$ be the natural projection. Because $p \circ f: \Omega \to \Delta/\langle\alpha\rangle$ is Γ-invariant, the map

$p \circ f \circ \pi^{-1} : \Delta \backslash K \to \Delta / \langle \alpha \rangle$ is a well-defined bounded analytic function. Since K has analytic capacity zero, $p \circ f \circ \pi^{-1}$ is extended to a holomorphic function $g : \Delta \to \Delta/\langle \alpha \rangle$. Then since the map p is a universal covering, there exists a lifting $h : \Delta \to \Delta$ of g such that $g = p \circ h$. Thus, $p \circ f = p \circ h \circ \pi$ on Ω. Hence we have $f = \varepsilon \circ h \circ \pi$ for some $\varepsilon \in \langle \alpha \rangle$, showing that f is Γ-invariant and the claim is proved.

Since f is Γ-invariant, $f \circ \pi^{-1} : \Delta \backslash K \to \Delta$ is a well-defined bounded analytic function and hence is extended to a holomorphic function $\rho : \Delta \to \Delta$. Because $f = \rho \circ \pi$ and $f(\infty) = \pi(\infty) = 0$, we have $\rho(0) = 0$ and $f'(\infty) = \rho'(0)\pi'(\infty)$. By extremality of the Ahlfors function, we have $\rho'(0) \geq 1$ since $f'(\infty) > 0$ and $\pi'(\infty) > 0$. Then Schwarz lemma implies that $\rho(z) = z$ and hence $f = \pi$. □

4. Main results

Theorem 1. *Let Ω be the Denjoy domain of type (K, π), and let Γ be the cover transformation group of the covering π. If f is the Ahlfors function for Ω and ∞, then the following are equivalent.*

(1) *f is Γ-invariant, i.e., $f \circ \gamma = f$ for all $\gamma \in \Gamma$;*
(2) *$f = \pi$;*
(3) *f is a covering onto its image;*
(4) *$f(\Omega) = \Delta \backslash K$;*
(5) *$K \backslash \mathbb{R}$ has logarithmic capacity zero.*

Moreover, if Ω is not simply connected, then each of the above conditions is equivalent to the following.

(6) *$f(\Omega) \neq \Delta$.*

Proof. First we show that (1) implies (2). If (1) holds, then the same reasoning as in the last part of the proof of Lemma 7 shows that $f = \pi$, and we see that condition (2) holds.

That (2) implies (3) is trivial.

Next assume that (3) holds. If $f(\Omega) = \Delta$ and f is a covering, then since Δ is simply connected, Ω is also simply connected. Lemma 4 implies that $K = \varnothing$, and hence (4) holds. On the other hand, if $f(\Omega)$ is a proper subset of Δ, then Lemma 7 shows that (4) holds. Hence (3) implies (4).

Assume that (4) holds. To prove (5) we may assume that $K \neq \varnothing$. Then from Lemma 7 condition (4) implies that $f = \pi$ and so f is Γ-invariant. We show that if f is Γ-invariant then (5) holds. Since $E = \partial\Omega$ is Γ-invariant, one verifies easily that the harmonic measure ω_E is Γ-invariant. From Lemma 2 we see that the conjugate harmonic function ω_E^* is also Γ-invariant. Since $g = \omega_E + i\omega_E^*$ is Γ-invariant, there exists a holomorphic function h defined on $\Delta \backslash K$ such that $g = h \circ \pi$ with $0 < \operatorname{Re} h < 1$ on $\Delta_- \backslash K$. Since by definition the analytic capacity of K is zero, considering the bounded analytic function $(h-1)/(h+1)$, we may assume by analytic continuation that h is holomorphic on Δ_-. Set $\omega_0 = \operatorname{Re} h$. Then ω_0 is harmonic on Δ_- and satisfies an identity $\omega_E = \omega_0 \circ \pi$. By definition the nontangential boundary value of $\omega_E|_H$ is 0 or 1 a.e. on the real axis. On the other hand, it follows from maximum principle for harmonic functions that $0 < \omega_0 < 1$ for all $z \in \Delta_-$. Hence we find that almost all nontangential boundary values of $\pi|_H$ belong to $\partial\Delta_-$. It is elementary to construct a conformal mapping $k : \Delta_- \to \Delta$. Then we conclude that $k \circ \pi : H \to$

Δ is an inner function; i.e., the absolute value of its nontangential boundary values is a.e. equal to one. Frostman's theorem [5, p. 79] implies that the image of any inner function covers the unit disk with the exception of a set of logarithmic capacity zero. Thus the set $k(K \cap \Delta_-)$ has logarithmic capacity zero. This in turn implies by the conformal invariance of vanishing of logarithmic capacity [11, p. 184] that $K \cap \Delta_-$ also has logarithmic capacity zero. Since $K \setminus \mathbb{R} = (K \cap \Delta_-) \cup \overline{(K \cap \Delta_-)}$, by using a fact that a countable union of sets of logarithmic capacity zero is of logarithmic capacity zero [12, p. 57], we see that (5) holds.

We proceed to show that if $K \cap \Delta_-$ has logarithmic capacity zero then f is Γ-invariant. In view of Lemma 2, it suffices to show that the conjugate harmonic function ω_E^* is Γ-invariant since ω_E is Γ-invariant. As before set $\omega_E = \omega_0 \circ \pi$ where ω_0 is a bounded harmonic function on $\Delta_- \setminus K$. The assumption implies that ω_0 is extended to a bounded harmonic function on Δ_- [12, p. 78]. Since any Γ-period of ω_E^* is a flux of ω_0 along some closed curve in $\Delta_- \setminus K$, all periods must vanish. Thus, ω_E^* is Γ-invariant.

Finally, the last assertion is clear from Lemmas 4 and 7. This completes the proof of Theorem 1. □

Corollary 1. *Let K be a subset of Δ. Then the following are equivalent.*

(1) *K is the omitted set of the Ahlfors function F for some maximal Denjoy domain and ∞ that is covering onto its image;*

(2) *$K \in \Sigma$ and $K \setminus \mathbb{R}$ has logarithmic capacity zero.*

Proof. If condition (1) holds, then by Lemma 6 the domain of definition of F is a Denjoy domain Ω of type (K', π) for some $K' \in \Sigma$ and π. Theorem 1 then implies that $f(\Omega) = \Delta \setminus K'$. Thus $K = K'$. Hence by Theorem 1 the set K satisfies (2).

Conversely, if K satisfies (2), let Ω be the Denjoy domain of type (K, π) for some covering π. Again, Theorem 1 shows that if $K \setminus \mathbb{R}$ has logarithmic capacity zero, then K is the omitted set of the Ahlfors function π for Ω and ∞. □

Corollary 2. *There exists a maximal Denjoy domain Ω such that the omitted set of the Ahlfors function for Ω and ∞ has positive logarithmic capacity.*

Proof. Let $S \subset (0, 1)$ be a compact set with zero linear measure but with positive logarithmic capacity. Using the Cantor ternary set on $[0, 1]$, we easily obtain such a set. Take a countable dense subset $\{a_n\}_{n=1}^{\infty}$ of S. Let

$$K = S \cup \{z | z = a_n \pm i/k \ (n, k \in \mathbb{N},\ k \geq n) \text{ and } |z| < 1\}.$$

Then it is clear that $K \in \Sigma$ and that $K \setminus \mathbb{R}$ has zero logarithmic capacity. Corollary 1 implies that K is an omitted set with positive logarithmic capacity. □

Let f be a bounded analytic function on a hyperbolic domain D. Then f is called an *inner* function if and only if $f \circ \phi$ is a usual inner function on Δ where $\phi: \Delta \to D$ denotes a holomorphic universal covering of D. Frostman's theorem [5, p. 79] implies that if a function is inner then its omitted set has logarithmic capacity zero. Corollary 2 immediately gives the following result.

Corollary 3. *There exists a maximal Denjoy domain Ω such that the Ahlfors function for Ω and ∞ is not an inner function.*

REFERENCES

1. S. D. Fisher, *On Schwarz's lemma and inner functions*, Trans. Amer. Math. Soc. **138** (1969), 229–240.
2. ——, *The moduli of extremal functions*, Michigan Math. J. **19** (1972), 179–183.
3. T. Gamelin, *Lectures on* $H^\infty(D)$, Notas Mat., La Plata, Argentina, Vol. 21, 1972.
4. J. B. Garnett, *Analytic capacity and measure*, Springer-Verlag, Berlin, Heidelberg, and New York, 1972.
5. ——, *Bounded analytic functions*, Academic Press, New York, 1981.
6. S. Ya. Havinson, *Analytic capacity of sets, joint nontriviality of various classes of analytic functions and the Schwarz lemma in arbitrary domains*, Amer. Math. Soc. Transl. Ser. 2, vol. 43, Amer. Math. Soc., Providence, RI, 1964, pp. 215–266.
7. C. D. Minda, *The image of the Ahlfors function*, Proc. Amer. Math. Soc. **83** (1981), 751–756.
8. Ch. Pommerenke, *Über die analytische Kapazität*, Arch. Math. (Basel) **11** (1960), 270–277.
9. E. Roding, *Über die Wertannahme der Ahlfors funktion in beliebigen Gebieten*, Manuscripta Math. **20** (1977), 133–140.
10. W. Rudin, *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc. **78** (1955), 333–342.
11. L. Sario and K. Oikawa, *Capacity functions*, Grundlehren Math. Wiss., vol. 149, Springer-Verlag, Berlin, Heidelberg, and New York, 1969.
12. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.
13. A. Yamada, *A remark on the image of the Ahlfors function*, Proc. Amer. Math. Soc. **88** (1983), 639–642.

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