PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 3, July 1992

A NOTE ON HYPERELLIPTIC CURVES

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(Communicated by Frederick R. Cohen)

ABSTRACT. There is no universal hyperelliptic curve over the space of $(2g+2)$-tuples in the projective line.

1. INTRODUCTION

If C is a Riemann surface of genus 1 or a hyperelliptic Riemann surface of genus $g \geq 2$, then C admits a g_2^1, i.e., a map $p\colon C \to \mathbf{P}^1$ of degree 2, branched over $2g+2$ points. The g_2^1 is unique up to the action of $\mathrm{PGL}_2\mathbf{C}$ on $\mathbf{P}^1$ and (if $g = 1$) translation in C. We may consider the space X_{2g+2} of sets of $2g+2$ (unordered) distinct points in $\mathbf{P}^1$—thus X_{2g+2} is an open set in the symmetric power $\mathrm{Sym}^{2g+2}\mathbf{P}^1 = \mathbf{P}^{2g+2}$—and the tautologous bundle and subbundle $(\mathbf{P}^1 \times X_{2g+2}, R_{2g+2})$, where the fiber of R_{2g+2} over the element $\{p_1, \ldots, p_{2g+2}\}$ of X_{2g+2} is the subset $\{p_1, \ldots, p_{2g+2}\}$ of $\mathbf{P}^1$, and ask whether there is a family $p_Y\colon Y_g \to X_{2g+2}$ of hyperelliptic curves of genus g over X_{2g+2}, admitting a $g_2^1 p\colon Y_g \to \mathbf{P}^1 \times X_{2g+2}$ over X_{2g+2} (i.e., such that $p_2 \circ p = p_Y$, where $Z_{2g+2} = \mathbf{P}^1 \times X_{2g+2}$ and $p_2\colon Z_{2g+2} \to X_{2g+2}$ is the projection) that is branched over R_{2g+2}.

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Theorem 1. *For $g \geq 0$, there does not exist a family $p\colon Y_g \to Z_{2g+2}$ satisfying the conditions above.*

The referee has pointed out a simple proof for g odd; our proof does not distinguish the cases g odd and g even (cf. Remark 4).

Proof. First we point out where the obstruction to the existence of Y_g lies. On a fiber of $Z_{2g+2} - R_{2g+2}$, there is a unique element $\alpha \in H^1(\mathbf{P}^1 - \{p_1, \ldots, p_{2g+2}\}, \mathbf{Z}/2)$ that is the Alexander dual of $\{p_1, \ldots, p_{2g+2}\}$ in $\mathbf{P}^1 = S^2$. On each fiber α defines the desired double branched covering and α is left invariant by the monodromy of $\pi_1 X_{2g+2}$ on $H^1(\mathbf{P}^1 - \{p_1, \ldots, p_{2g+2}\}, \mathbf{Z}/2)$. There exists a $\mathbf{Z}/2$-cohomology class on $Z_{2g+2} - R_{2g+2}$ that restricts to an α

Received by the editors November 20, 1990.
1991 *Mathematics Subject Classification*. Primary 57M12; Secondary 14H15.
Key words and phrases. Hyperelliptic curve, braid group.
This work was partially supported by the N.S.F.

on a fiber if and only if the differential $d_2\colon H^1_{\text{inv}}(\mathbf{P}^1 - \{p_1, \dots, p_{2g+2}\}, \mathbf{Z}/2) \to H^2(X_{2g+2}, \mathbf{Z}/2)$ in the E_2 term of the spectral sequence for the fibering t satisfies $d_2(\alpha) = 0$. Otherwise $d_2(\alpha)$ is the obstruction to the existence of Y_g.

The case $g = 0$ is degenerate, because the mapping class group of a sphere is trivial. X_2 contains the space $\mathbf{R}P^2$ of antipodal pairs as a deformation retract. Over $\mathbf{R}P^2$, $Z_2 - R_2$ deformation retracts fiberwise to a bundle of great circles, which may be identified with the unit tangent bundle $L(4, 1)$ of $\mathbf{R}P^2$. So there is no double cover of $Z_2 - R_2$ that restricts to a nontrivial covering on each fiber.

Now we assume $g \geq 1$. By definition $\pi_1 X_{2g+2}$ is the braid group $B(2g+2, S^2)$. Adding the relator $t_1 \cdots t_n^2 \cdots t_1$ to the standard presentation $(t_1, \dots, t_n\colon t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},\ [t_i, t_j]\ (|i-j| > 1))$ of Artin's braid group $B(n+1, \mathbf{R}^2)$ gives a presentation of $B(n+1, S^2)$. To see this suppose x is in the kernel of the natural homomorphism from $B(n+1, \mathbf{R}^2)$ to $B(n+1, S^2)$. x can be represented by a family of unordered $(n+1)$-tuples in $\mathbf{R}^2$, parametrized by a loop. The null homotopy in X_{n+1} can be realized by a map of a disc to X_{n+1} that restricts on the boundary to the given loop. Take the disc to be transverse to the set of $(n+1)$-tuples containing ∞. Then there are finitely many points in the disc, where one of the points in the $(n+1)$-tuple becomes ∞ and the monodromy in $B(n+1, \mathbf{R}^2)$ of a small loop around such a point is conjugate to $(t_1 \cdots t_n^2 \cdots t_1)^{\pm 1}$ in $B(n+1, \mathbf{R}^2)$, as $t_1 \cdots t_n^2 \cdots t_1$ can be represented by holding all but the first point fixed and moving the first point around a circle containing the other n points in its interior.

The kernel of the homomorphism $h\colon B(2g+2, S^2) \to \Gamma_0^{2g+2}$ is the image of $\mathbf{Z}/2 = \pi_1 \mathrm{SO}(3)$ in $\pi_1 X_{2g+2}$ given by the action of $\mathrm{SO}(3)$ on S^2; it is generated by $(t_1 \cdots t_{2g+1})^{2g+2}$, as the braid $(t_1 \cdots t_{2g+1})$ can be represented by a rotation of the sphere through an angle $2\pi/2g+2$.

Now we consider whether the homomorphism $h\colon B(2g+2, S^2) \to \Gamma_0^{2g+2}$ lifts to a homomorphism $\overline{h}\colon B(2g+2, S^2) \to \Gamma_g$ such that $h = \pi \circ \overline{h}$ where $\pi\colon Z(i) \to \Gamma_0^{2g+2}$ is the homomorphism from the centralizer $Z(i)$ of a hyperelliptic involution i to the mapping class group Γ_0^{2g+2}. $Z(i)$ is generated by Dehn twists $\overline{t}_1, \dots, \overline{t}_{2g+1}$ on curves $C_1, \dots, C_{2g+1}$ that are invariant under the hyperelliptic involution i. $\pi(\overline{t}_i) = [t_i] \in T_0^{2g+2}$, the image of $t_i \in B(2g+2, S^2)$. If $\overline{h}$ exists, then $\overline{h}(t_i) = \overline{t}_i$ or $\overline{t}_i \circ i$. Since the elements t_i are conjugate in $B(2g+2, S^2)$, the elements $\overline{h}(t_i)$ are all conjugate in Γ_2; so either $\overline{h}(t_i) = t_i$ for all i or $\overline{h}(t_i) = \overline{t}_i \circ i$ for all i. In either case $\overline{h}(t_1 \cdots t_{2g+1}^2 \cdots t_1) = \overline{t}_1 \cdots \overline{t}_{2g+1}^2 \cdots \overline{t}_1$ as i is central and the number $4g+2$ of letters in $t_1 \cdots t_{2g+1}^2 \cdots t_1$ is even. But $\overline{t}_1 \cdots \overline{t}_{2g+1}^2 \cdots \overline{t}_1 = i$ in Γ_2. To see this draw the curves $C_1, C_1 \cdot \overline{t}_1, \dots, C_1 \overline{t}_1 \cdots \overline{t}_{2g+1}^2 \cdots \overline{t}_1$ and observe that the orientation of C_i is reversed. Alternatively, consider the curves $H_\theta\colon y^2 = (x-a_1) \cdots (x-a_{2g+1})(x - Re^{i\theta})$ with $R > |a_1|, \dots, |a_{2g+1}|$. Fix x with $|x| < R$. After increasing θ by 2π the sign of y has changed.

Since h does not lift to $\overline{h}$, there does not exist a family Y_g.

Remarks. (1) It follows from $B(2g+2, S^2) = B(2g+2, \mathbf{R}^2)/(t_1 \cdots t_{2g+1}^2 \cdots t_1)$ that $H_1(B(2g+2, S^2), \mathbf{Z}) = \mathbf{Z}/(4g+2)$. Another way to see this is to consider

n points with homogeneous coordinates (α_1, β_1) in $\mathbf{P}^1$ and write the discriminant $\prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^2 = F(\Sigma_0, \ldots, \Sigma_n)$ where $\Sigma_i(\alpha_i, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \sigma_i(\alpha_1/\beta_1, \ldots, \alpha_n/\beta_n)\beta_1 \cdots \beta_n$ and σ_i is the ith elementary symmetric function. Then each Σ_i has degree n so F is a homogeneous polynomial of degree $2(n-1)$. Since F is irreducible, the complement of the locus $F = 0$ in $\mathbf{P}^n$ has first homology group $\mathbf{Z}/2(n-1)$. Since Γ_0^{2g+2} is obtained from $B(2g+2, S^2)$ by adjoining the relation $(t_1 \cdots t_{2g+1})^{2g+2}$, it follows that $H_1(\Gamma_0^{2g+2}, \mathbf{Z}) = \mathbf{Z}/(4g+2)$. O. Gabber showed me this argument.

(2) Now suppose $g \geq 2$. Consider the group $\widetilde{G} = \langle t_1, \ldots, t_{2g+1} : t_it_{i+1}t_i = t_{i+1}t_it_{i+1}\ \ (1 \leq i \leq 2g),\ [t_i, t_j]\ \ (|i-j| > 1),\ [t_1, t_1 \cdots t_{2g+1}^2 \cdots t_1]\rangle$. Then the center of $\widetilde{G}$ is $\mathbf{Z}/2 \oplus \mathbf{Z}$, with generators

$$A = (t_1 \cdots t_{2g+1})^{2g+2}(t_1 \cdots t_{2g+1}^2 \cdots t_1)^{-(g+1)}, \qquad I = (t_1 \cdots t_{2g+1}^2 \cdots t_1).$$

The kernels of the natural maps from $\widetilde{G}$ to $B(2g+2, S^2)$ and $Z(i)$ are I and $\langle A, I^2\rangle$, respectively. So $\widetilde{G}/\langle I^2\rangle$ is the pullback of the two central extensions $B(2g+2, S^2)$, $Z(i)$ of Γ_0^{2g+2}. Since $H_1(\widetilde{G}/\langle I^2\rangle, \mathbf{Z}) = \mathbf{Z}/8g+4$, we see that the two extensions $B(2g+2, S^2)$, $Z(i)$ of Γ_0^{2g+2} arise from elements of $H^2(\Gamma_0^{2g+2}, \mathbf{Z}/2)$ with the same image in $\operatorname{Hom}(H_2(\Gamma_0^{2g+2}, \mathbf{Z}/2), \mathbf{Z}/2)$, differing by the nontrivial element of $\operatorname{Ext}(H_1(\Gamma_0^{2g+2}, \mathbf{Z}/2), \mathbf{Z}/2)$.

(We assumed $g \geq 2$ because if $g = 1$, $\Gamma_0^4 = (\mathbf{Z}/2 * \mathbf{Z}/3) \ltimes (\mathbf{Z}/2)^2$ and the central extension corresponding to $Z(i)$ is not $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})$ but rather $\mathrm{SL}(2, \mathbf{Z}) \ltimes (\mathbf{Z}/2)^2$.)

(3) In fact $H_2(B(n, S^2)) = \mathbf{Z}/2$. One way to see this is to use Arnold's calculation [A] of $H_*(B(n, \mathbf{R}^2), \mathbf{Z})$ to see that $H_2(B(n, \mathbf{R}^2), \mathbf{Z}) \cong \mathbf{Z}/2$, $n \geq 4$. Since $B(n, S^2)$ is obtained from $B(n, \mathbf{R}^2)$ by adding a relator $t_1 \cdots t_{n-1}^2 \cdots t_1$ that has infinite order in homology, Hopf's formula [Br] shows that $H_2(B(n, \mathbf{R}^2)) \cong H_2(B(n, S^2))$.

The 5-term exact sequence for $PB(n, \mathbf{R}^2) \to B(n, \mathbf{R}^2) \to \Sigma_n$ shows that the natural map $H_2(B(n, \mathbf{R}^2)) \to H_2(\Sigma_n, \mathbf{Z})$ is onto. Since this map factors through $H_2(\Gamma_0^n, \mathbf{Z})$, the exact sequence $H_2(B(n, S^2), \mathbf{Z}) \to H_2(\Gamma_0^n, \mathbf{Z}) \to \mathbf{Z}/2 \to 0$ from the 5-term sequence for $\mathbf{Z}/2 \to B(n, S^2) \to \Gamma_0^n$ is split: $H_2(\Gamma_0^n, \mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$. It follows that $H_2(\Gamma_2, \mathbf{Z}) \cong \mathbf{Z}/2$, which is confirmed by [C2, BCT].

(4) Since $B(2g+2, S^2)$ and Γ_0^{2g+2} have abelianization $\mathbf{Z}/(4g+2)$ while $Z(i)$ has abelianization $\mathbf{Z}/(4g+2)$ for g even and $\mathbf{Z}/(8g+4)$ for g odd, there can be no lift $\overline{h}$ of the homomorphism h in case g is odd, so for odd g the theorem becomes more elementary.

Finally let us remark that if we consider the double cover $\widetilde{X}_{2g+2}$ of X_{2g+2}, with fundamental group an extension of A_{2g+2} by the pure braid group $PB(2g+2, S^2)$, the obstruction vanishes because $d_2(\alpha)$ is the Bockstein of the nontrivial element of $H^1(B(2g+2, S^2), \mathbf{Z}/2)$, and the Bockstein homomorphism is natural. If $q = 1$, then one can alternatively consider the cover X_4' of X_4 corresponding to the subgroup S_3 of the quotient S_4 defined by permuting the points; $\pi_1 X_4' = \mathrm{SL}_2\mathbf{Z}$.

The homomorphism $\mathrm{SL}_2\mathbf{Z} \to \Gamma_0^4 = \mathrm{PSL}_2\mathbf{Z} \ltimes (\mathbf{Z}/2)^2$ lifts to $\mathrm{SL}_2\mathbf{Z} \ltimes (\mathbf{Z}/2)^2$. Since X_{2g+2} has the homotopy type of a $\mathrm{SL}_2\mathbf{C}$ bundle over a

$K(B(2g+2, S^2), 1)$, there can be no obstruction to constructing the required family of genus 1 curves. As a check, the family can be exhibited

(1)
$$Z^2 = \frac{(QX-PY)(AX^3+BX^2Y+CXY^2+DY^3)(\alpha_1 Q-\beta_1 P)^3(\alpha_2 Q-\beta_2 P)^3(\alpha_3 Q-\beta_3 P)^3}{(\alpha_1\beta_2-\beta_1\alpha_2)^2(\alpha_1\beta_3-\beta_1\alpha_3)^2(\alpha_2\beta_3-\alpha_3\beta_2)^2},$$

where the 4 points have projective coordinates $(\alpha_1 : \beta_1)$, $(\alpha_2 : \beta_2)$, $(\alpha_3 : \beta_3)$, $(P : Q)$, the last being the coordinates of the distinguished element of the 4-tuple, and

$$AX^3 + \cdots + DY^3 = (\beta_1 X - \alpha_1 Y)(\beta_2 X - \alpha_2 Y)(\beta_3 X - \alpha_3 Y).$$

Note that replacing (α_i, β_i) by $(\lambda_i\alpha_i, \lambda_i\beta_i)$ leaves Z unchanged, while replacing (P, Q) by $(\lambda P, \lambda Q)$ changes Z to $\lambda^5 Z$. So the family of genus 1 curves lies in the $\mathbf{P}^1$ bundle over X_4' that is the pullback, by the projection $X_4' \to \mathbf{P}^1$, of the rational normal scroll $\Sigma_5 \to \mathbf{P}^1$, which is obtained from the degree 5 line bundle over $\mathbf{P}^1$ by completing each fiber $\mathbf{C}$ to a projective line.

More generally, if we consider the cover X_{2g+2}' of X_{2g+2} corresponding to the subgroup $S_{2g+1} \subset S_{2g+2}$ stabilizing one point, the restriction r^*y of the nontrivial element $y \in H^1(B(2g+2, S^2), \mathbf{Z}/2)$ becomes the reduction of a mod 4 class, so $\beta r^* y = 0$. The family of curves can be constructed analogously to (1) in a $\mathbf{P}^1$ bundle over X_{2g+2} pulled back from the rational normal scroll Σ_{4g^2+g}.

Acknowledgment

The question addressed arose during interesting discussions with F. R. Cohen and R. Hain. There is some overlap with [C1, C2, BCT]. I would like to thank O. Gabber who pointed out a blunder, and the I.H.E.S. and the I.A.S. for their hospitality and support.

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