K-THEORETICAL INDEX THEOREMS FOR GOOD ORBIFOLDS

CARLA FARSI

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Abstract. In this note we study index theory for general and good orbifolds. We prove a $K$-theoretical index theorem for good orbifolds, and from this we deduce as a corollary a numerical index formula.

Let $D$ be a pseudodifferential elliptic operator on the closed orbifold $Q$. In §1 we give an index formula involving a certain class $[\tilde{D}]$ associated to $D$. In §2 we prove a $K$-theoretical index theorem (in analogy with the main theorem in [9]) for good orbifolds (a good orbifold is an orbifold that can be covered by a smooth manifold), which relates the class $[\tilde{D}]$ with the class of its symbol. It is also natural to consider in this case, besides the usual (analytical) index of $D$, its Atiyah-Singer index [2, 12]. We then recover from the $K$-theoretical index theorem the main theorem in [2]. In §3 we relate the analytical and the Atiyah-Singer indices of $D$.

1. AN ORBIFOLD INDEX THEOREM

We first give an index theorem for general orbifolds $Q$. We can suppose throughout this paper that $Q$ is even-dimensional, since the case in where the dimension of $Q$ is odd can be obtained from this by crossing with $T$. All the orbifolds are also assumed to be orientable and closed. Note that any nonorientable orbifold is always finitely covered by an orientable orbifold.

Every orbifold $Q$ arises as a quotient $Q = P/G$, where $G$ is a compact group acting locally freely on the smooth manifold $P$ [11]. In our case we can choose $G = \text{SO}(q)$, where $q$ is the dimension of $Q$ and $P$ is the orthonormal frame bundle of $Q$.

Let $\eta^{(0)}$ and $\eta^{(1)}$ be two orbifold vector bundles over $Q$. We say that $D$ is an elliptic pseudodifferential operator on $Q$ acting from the $L^2$-sections $\Gamma(\eta^{(0)})$ of the bundle $\eta^{(0)}$ to the $L^2$-sections $\Gamma(\eta^{(1)})$ of $\eta^{(1)}$ if on each orbifold chart $U_i \approx \mathbb{R}^q/G_i$ the lift of $D$ to $\mathbb{R}^q$ is a pseudodifferential elliptic operator. We assume throughout this paper that $D$ has order 0. (We can always reduce to this case.) In analogy with the manifold case, every section of $\eta^{(0)}$ that is in $\text{Ker}(D)$ is $C^\infty$ and so is every section of $\eta^{(1)}$ in $\text{Ker}(D^*)$. This is because we only use local properties of $D$. Also $\text{Ker}(D)$ and $\text{Coker}(D)$ are finite-dimensional, so

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that [12]

\[
\text{Ind}_a(D) = \dim(\ker(D)) - \dim(\operatorname{coker}(D)).
\]

We define \( C(P) \times G \) to be the orbifold \( C^\ast \)-algebra \( C^\ast(Q) \) [6]. The element \([\tilde{D}] \in KK(C^\ast(Q), C)\), defined in [6], coincides with the element associated to the lift \( \tilde{D} \) of \( D \) to \( P \). Therefore we can consider the image of \([\tilde{D}]\) in the cyclic cohomology group \( H^c\operatorname{ev}(C^\infty(P \times G)) \) via \( \chi^\ast \). As remarked by Connes at the end of §8 in [5], \( C^\infty(G) \) embeds in \( C^\infty(P \times G) : C^\infty(G) \xrightarrow{i} C^\infty(P \times G) \) and the restriction \( r^\ast \chi^\ast([\tilde{D}]) \) of \( \chi^\ast([\tilde{D}]) \) to \( C^\infty(G) \) is given exactly by \( r^\ast \chi^\ast([\tilde{D}]) = S^l \chi, q = 2l = \dim Q \), where \( \chi \) is the distributional index character of \( \tilde{D} \) defined by Atiyah in [1], i.e., \( \chi \in HC^0(C^\infty(G)) \),

\[
\chi(f) = \text{Tr} \left( \text{action of } f \text{ in } \ker \tilde{D} \right) - \text{Tr} \left( \text{action of } f \text{ in } \ker D^\ast \right)
\]

and \( S \) is an operator defined by Connes in [5]. Therefore,

**Theorem 1.** Let \( D \) be a pseudodifferential elliptic operator on the spin\(^c\) orbifold \( Q \). Then

\[
\text{Ind}_a(D) = \langle \chi^\ast([\tilde{D}]), r^\ast(1) \rangle,
\]

where \( r : C^\infty(G) \hookrightarrow C^\infty(P \times G) \) is the canonical embedding, \( 1 \in HC^0(C^\infty(G)) \) is the element corresponding to the constant function 1, and \( r^\ast : HC^0(C^\infty(G)) \to HC^0(C^\infty(P \times G)) \) is the induced homomorphism.

**Proof.** \( \text{Ind}_a(D) = \langle \chi, 1 \rangle, \chi \in HC^0(C^\infty(G)), \quad 1 \in HC^0(C^\infty(G)) \) by [11]. Note that 1 corresponds to the function 1 \( \in C^\infty(G) \) by [4]. Because \( \langle \, , \, \rangle \) is \( S \)-invariant [5],

\[
\text{Ind}_a(D) = \langle r^\ast \chi^\ast([\tilde{D}]), 1 \rangle = \langle \chi^\ast([\tilde{D}]), r^\ast(1) \rangle.
\]

When \( G \) acts freely on \( P \) this is the Atiyah-Singer index theorem [3] (c.f. [5, §6, Theorem 5]).

**2. Good orbifolds**

In the case of a good orbifold (i.e., its universal cover is a smooth manifold) another definition of index for a pseudodifferential elliptic operator \( D \) is possible (see [2, 12]). In fact, let \( \hat{Q} \) be the universal cover of \( Q \), \( \pi_1^{\text{orb}}(Q) = \Gamma \) be the fundamental group of \( Q \), and \( \hat{D} \) be the lift of \( D \) to \( \hat{Q} \). Then \( \hat{Q}/\Gamma = Q \) and \( \Gamma \) acts on \( \hat{Q} \) properly. The Atiyah-Singer index of \( D \), AS-index, is defined as follows (c.f. [2]). On \( \hat{Q} \) we consider a \( \Gamma \)-invariant positive measure \( d\hat{\mu} \) (the lift of a positive measure \( d\mu \) on \( Q \)). Let \( \hat{\eta}(i), i = 0, 1 \), be the bundle over \( \hat{Q} \) lift of the bundle \( \eta(i), i = 0, 1 \) over \( Q \). Note also that \( L^2(\hat{\eta}(0) \oplus \hat{\eta}(1)) \cong L^2(\hat{Q}) \times \text{End}(C^n), n = \dim(\eta(0)) + \dim(\eta(1)) \), with the action of \( \Gamma \) trivial on \( \text{End}(C^n) \). The bounded operators on \( L^2(\hat{\eta}(0) \oplus \hat{\eta}(1)) \) that commute with the action of \( \Gamma \) form a von-Neumann algebra \( A(\hat{\eta}) \) that has a natural trace function denoted by \( \text{Tr}_\Gamma \). In particular if \( P \in A(\hat{\eta}) \) is an orthogonal projection onto a subspace \( H \) of \( L^2(\hat{\eta}(0) \oplus \hat{\eta}(1)) \), so that \( H \) is a \( \Gamma \)-module, we define

\[
\dim_\Gamma(H) = \text{Tr}_\Gamma(P) \in \mathbb{R}.
\]
Applying this to \( \text{Ker}(\hat{D}) \) and \( \text{Coker}(\hat{D}) \) we get a finite real-valued index

\[
\text{AS-ind}(D) = \dim_{\Gamma}(\text{Ker}(\hat{D}) - \dim_{\Gamma}(\text{Coker}(\hat{D})).
\]

Next we will define the \( K \)-theoretical \( \Gamma \)-index of \( D \). Firstly we will rewrite the orbifold, \( C^*\)-algebra \( C^*(Q) \), up to Morita equivalence.

**Proposition 2.** Let \( Q \) be a good orbifold. Then \( C_0(\hat{Q}) \rtimes \Gamma \) and \( C^*(Q) \) are Morita equivalent.

**Proof.** Let \( \hat{P} \) be the orthogonal frame bundle of \( \hat{Q} \). The following diagram

\[
P \xrightarrow{\text{SO}(q)} \hat{Q} \\
\Gamma \downarrow \quad \downarrow \Gamma \\
\hat{P} \xrightarrow{\text{SO}(q)} Q
\]

commutes. Since \( \Gamma \) and \( \text{SO}(q) \) act freely on \( \hat{P} \) and their actions commute, by a theorem of P. Green (see [15]), \( C^*(\hat{P}/\Gamma, \text{SO}(q)) \) is Morita equivalent to \( C^*(\hat{P}/\text{SO}(q), \Gamma) \). □

Hence an elliptic pseudodifferential elliptic operator \( D \) on \( Q \) determines a class

\[
[D] \in KK(C_0(\hat{Q}) \rtimes \Gamma, C) \cong K K_{\Gamma}(C_0(\hat{Q}), C).
\]

To define the \( K \)-theoretical \( \Gamma \)-index \( \text{IND}_{\Gamma}(\hat{D}) \) of \( D \), which is an element of \( K_0(C^*(\Gamma)) \), we first observe that since \( \Gamma \) acts on \( L^2(\eta^{(i)}) \) and \( L^2(\eta^{(1)}) \), and so it determines an element of \( K_0(C^*(\Gamma)) \), which we call \( \text{IND}_{\Gamma}(\hat{D}) \) (c.f. [10, §4]). \( \text{IND}_{\Gamma}(\hat{D}) \) is represented by the projections of \( L^2(\eta^{(0)}) \) onto \( \text{Ker}(\hat{D}) \) and of \( L^2(\eta^{(1)}) \) onto \( \text{Ker}(\hat{D}^* \Gamma) \). The following theorem can be recovered from a theorem of Kasparov [9].

**Theorem 3.** Let \( D \) be a pseudodifferential elliptic operator on the good orbifold \( Q, D: L^2(\eta^{(0)}) \to L^2(\eta^{(1)}) \). Let \( \hat{Q} \) be the universal cover of \( Q, \Gamma = \pi_1^{\text{ORB}}(Q), \) and \( \hat{D} \) be the operator \( D \) lifted to \( \hat{Q} \). Then,

\[
\text{IND}_{\Gamma}(\hat{D}) = \text{IND}_{\Gamma}(\hat{D}) \quad \text{in } KK(C, C^*(\Gamma)),
\]

with

\[
\text{IND}_{\Gamma}(\hat{D}) = [C] \bigotimes_{C_0(\hat{Q}) \rtimes \Gamma} j_\Gamma([\hat{D}]),
\]

where \([\hat{D}] \in KK_{\Gamma}(C_0(\hat{Q}), C)\), \( j_\Gamma: KK_{\Gamma}(A, B) \to KK(A \rtimes \Gamma, B \rtimes \Gamma) \) is the canonical homomorphism, and \([C] \in K_0(C_0(\hat{Q}) \rtimes \Gamma) \) is determined by the projection \( p(x, g) = \sqrt{c(x)c(g^{-1}x)} \), where \( c \in C_0^\infty(\hat{Q}) \) is such that \( \int_{\Gamma} c(xg) \, dg = 1 \) and where \( c \geq 0 \).

Note that we could also have an index theorem with coefficients in a \( C^* \)-bundle rather than in a vector bundle in the spirit of [14].

As a corollary to Theorem 3 we obtain.
Theorem 4. Let $D$, $Q$, $\hat{Q}$, $\Gamma$, and $\hat{D}$ be as in Theorem 3. Then,
\[ \tau(\text{IND}_a(\hat{D})) = \text{AS-ind}(D), \]
where $\tau$ is the canonical trace on $K_0(C^*(\Gamma)) = (\text{Idempotents of } C^*(\Gamma) \otimes \mathcal{H})/\sim$.

Proof. $\tau$ is given by $\tau(a \otimes A) = \tau^T(a) \otimes T(A)$ where $a \in C^*(\Gamma)$, $a = \sum_{g \in \Gamma} \lambda_g [g]$, $\lambda_g \in \mathbb{R}$, $\tau^T(a) = \lambda_e$, $e$ = unit of $\Gamma$, $A \in \mathcal{H} = \mathcal{H}(L^2(Q))$, $T$ = canonical trace on $\mathcal{H}(L^2(Q)) \subseteq \mathcal{B}(L^2(Q))$. This trace coincides with the trace in [2, p. 57]. □

3. Relations between indices

As we have seen in §1 and in §2, if $Q$ is a good orbifold and $D$ is a pseudodifferential elliptic operator on $Q$, then we can define the two indices $\text{Ind}_a(D)$ and $\text{AS-ind}(D)$. The first one is necessarily an integer, while the second one is a rational number. In general they do not coincide, but there is an interesting relation between them, which is a corollary of the main theorems in [13] and [2] (c.f. also [12, III]). In fact Atiyah's argument applies also to the case in where the action is not free.

Theorem 5. Let $D$, $Q$, $\hat{Q}$, $\Gamma$, and $\hat{D}$ be as in Theorem 3. Then,
\[ \text{Ind}_a(D) = \text{AS-ind}(D) + R, \]
where (with the notation as in the introduction in [13]),
\[ R = \sum_{\text{strata } Q} \int \frac{1}{m_i} \langle (-1)^{\Sigma_i} (\text{ch}_{\Sigma}(D) \mathcal{F}^{\Sigma}(Q)), [\Sigma_i] \rangle, \]
with $\Sigma_i$ running over the strata of $Q$.

For example, if $\mathcal{E}$ is the Euler operator on $Q$, then $\text{Ind}_a(\mathcal{E})$ is equal to the Euler characteristic of $Q$ as a vector space (c.f. [11, PROPOSITION]) and $\text{AS-ind}(\mathcal{E}) = X_{\Sigma}(Q)$, where $X_{\Sigma}(Q)$ is the Euler-Satake characteristic of $Q$, by the Gauss-Bonnet theorem for orbifolds of Satake [16] and the general formula in [12]. Since $R$ depends only on the singular structure of $Q$, it follows that $R$ is 0 if $Q$ is a smooth manifold, and so in that case we recover the main theorem in [2] from Theorem 5, $\text{Ind}_a(D) = \text{AS-ind}(D)$.

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References


Department of Mathematics, University of Colorado, Campus Box 426, Boulder, Colorado, 80309  
E-mail address: farsi@euclid.colorado.edu