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K-THEORETICAL INDEX THEOREMS FOR GOOD ORBIFOLDS

CARLA FARSI

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ABSTRACT. In this note we study index theory for general and good orbifolds. We prove a K-theoretical index theorem for good orbifolds, and from this we deduce as a corollary a numerical index formula.

Let D be a pseudodifferential elliptic operator on the closed orbifold Q. In §1 we give an index formula involving a certain class $[\widetilde{D}]$ associated to D. In §2 we prove a K-theoretical index theorem (in analogy with the main theorem in [9]) for good orbifolds (a good orbifold is an orbifold that can be covered by a smooth manifold), which relates the class $[\widetilde{D}]$ with the class of its symbol. It is also natural to consider in this case, besides the usual (analytical) index of D, its Atiyah-Singer index [2, 12]. We then recover from the K-theoretical index theorem the main theorem in [2]. In §3 we relate the analytical and the Atiyah-Singer indices of D.

1. AN ORBIFOLD INDEX THEOREM

We first give an index theorem for general orbifolds Q. We can suppose throughout this paper that Q is even-dimensional, since the case in where the dimension of Q is odd can be obtained from this by crossing with $\mathbf{T}$. All the orbifolds are also assumed to be orientable and closed. Note that any nonorientable orbifold is always finitely covered by an orientable orbifold.

Every orbifold Q arises as a quotient $Q = P/G$, where G is a compact group acting locally freely on the smooth manifold P [11]. In our case we can choose $G = \mathrm{SO}(q)$, where q is the dimension of Q and P is the orthonormal frame bundle of Q.

Let $\eta^{(0)}$ and $\eta^{(1)}$ be two orbifold vector bundles over Q. We say that D is an elliptic pseudodifferential operator on Q acting from the L^2-sections $\Gamma(\eta^{(0)})$ of the bundle $\eta^{(0)}$ to the L^2-sections $\Gamma(\eta^{(1)})$ of $\eta^{(1)}$ if on each orbifold chart $U_i \approx \mathbf{R}^q/G_i$ the lift of D to $\mathbf{R}^q$ is a pseudodifferential elliptic operator. We assume throughout this paper that D has order 0. (We can always reduce to this case.) In analogy with the manifold case, every section of $\eta^{(0)}$ that is in $\mathrm{Ker}(D)$ is C^∞ and so is every section of $\eta^{(1)}$ in $\mathrm{Ker}(D^*)$. This is because we only use local properties of D. Also $\mathrm{Ker}(D)$ and $\mathrm{Coker}(D)$ are finite-dimensional, so

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that [12]

$$\text{Ind}_a(D) \underset{\text{def}}{=} \text{Dim}(\text{Ker}(D)) - \text{Dim}(\text{Coker}(D)).$$

We define $C(P) \rtimes G$ to be the orbifold C^*-algebra $C^*(Q)$ [6]. The element $[\widetilde{D}] \in KK(C^*(Q), \mathbf{C})$, defined in [6], coincides with the element associated to the lift $\widetilde{D}$ of D to P. Therefore we can consider the image of $[\widetilde{D}]$ in the cyclic cohomology group $HC^{ev}(C^\infty(P \rtimes G))$ via ch^*. As remarked by Connes at the end of §8 in [5], $C^\infty(G)$ embeds in $C^\infty(P \rtimes G) : C^\infty(G) \overset{r}{\hookrightarrow} C^\infty(P \rtimes G)$ and the restriction $r^*ch^*([\widetilde{D}])$ of $ch^*([\widetilde{D}])$ to $C^\infty(G)$ is given exactly by $r^*ch^*([\widetilde{D}]) = S^l\chi$, $q = 2l = \dim Q$, where χ is the distributional index character of $\widetilde{D}$ defined by Atiyah in [1], i.e., $\chi \in HC^0(C^\infty(G))$,

$$\chi(f) \underset{\text{def}}{=} \text{Tr}\begin{pmatrix} \text{action of} \\ f \text{ in } \ker \widetilde{D} \end{pmatrix} - \text{Tr}\begin{pmatrix} \text{action of} \\ f \text{ in } \ker \widetilde{D}^* \end{pmatrix}$$

and S is an operator defined by Connes in [5]. Therefore,

Theorem 1. *Let D be a pseudodifferential elliptic operator on the* spin^c *orbifold Q. Then*

$$\text{Ind}_a(D) = \langle ch^*[\widetilde{D}], r_*(\mathbf{1}) \rangle,$$

where $r : C^\infty(G) \hookrightarrow C^\infty(P \rtimes G)$ is the canonical embedding, $\mathbf{1} \in HC_{ev}(C^\infty(G))$ is the element corresponding to the constant function 1, and $r_* : HC_{ev}(C^\infty(G)) \to HC_{ev}(C^\infty(P \rtimes G))$ is the induced homomorphism.

Proof. $\text{Ind}_a(D) = \langle \chi, \mathbf{1} \rangle$, $\chi \in HC^0(C^\infty(G))$, $\mathbf{1} \in HC_{ev}(C^\infty(G))$ by [11]. Note that $\mathbf{1}$ corresponds to the function $1 \in C^\infty(G)^G \equiv HC_{ev}(C^\infty(G))$ by [4]. Because $\langle\ ,\ \rangle$ is S-invariant [5],

$$\text{Ind}_a(D) = \langle r^*ch^*[\widetilde{D}], \mathbf{1} \rangle = \langle ch^*[\widetilde{D}], r_*(\mathbf{1}) \rangle. \quad \square$$

When G acts freely on P this is the Atiyah-Singer index theorem [3] (c.f. [5, §6, Theorem 5]).

2. Good orbifolds

In the case of a good orbifold (i.e., its universal cover is a smooth manifold) another definition of index for a pseudodifferential elliptic operator D is possible (see [2, 12]). In fact, let $\widehat{Q}$ be the universal cover of Q, $\pi_1^{\text{ORB}}(Q) = \Gamma$ be the fundamental group of Q, and $\widehat{D}$ be the lift of D to $\widehat{Q}$. Then $\widehat{Q}/\Gamma = Q$ and Γ acts on $\widehat{Q}$ properly. The Atiyah-Singer index of D, AS-index, is defined as follows (c.f. [2]). On $\widehat{Q}$ we consider a Γ-invariant positive measure $d\hat{\mu}$ (the lift of a positive measure $d\mu$ on Q). Let $\hat{\eta}^{(i)}$, $i = 0, 1$, be the bundle over $\widehat{Q}$ lift of the bundle $\eta^{(i)}$, $i = 0, 1$ over Q. Note also that $L^2(\hat{\eta}^{(0)} \oplus \hat{\eta}^{(1)}) \cong L^2(\widehat{Q}) \times \text{End}(\mathbf{C}^n)$, $n = \text{Dim}(\hat{\eta}^{(0)}) + \text{Dim}(\hat{\eta}^{(1)})$, with the action of Γ trivial on $\text{End}(\mathbf{C}^n)$. The bounded operators on $L^2(\hat{\eta}^{(0)} \oplus \hat{\eta}^{(1)})$ that commute with the action of Γ form a von-Neumann algebra $A(\hat{\eta})$ that has a natural trace function denoted by Tr_Γ. In particular if $P \in A(\hat{\eta})$ is an orthogonal projection onto a subspace H of $L^2(\hat{\eta}^{(0)} \oplus \hat{\eta}^{(1)})$, so that H is a Γ module, we define

$$\text{Dim}_\Gamma(H) \underset{\text{def}}{=} \text{Tr}_\Gamma(P) \in \mathbf{R}.$$

Applying this to $\operatorname{Ker}(\widehat{D})$ and $\operatorname{Coker}(\widehat{D})$ we get a finite real-valued index

$$\text{AS-ind}(D) \underset{\text{def}}{=} \operatorname{Dim}_\Gamma(\operatorname{Ker}(\widehat{D}) - \operatorname{Dim}_\Gamma(\operatorname{Coker}(\widehat{D})).$$

Next we will define the K-theoretical Γ-index of D. Firstly we will rewrite the orbifold, C^*-algebra $C^*(Q)$, up to Morita equivalence.

Proposition 2. *Let Q be a good orbifold. Then $C_0(\widehat{Q}) \rtimes \Gamma$ and $C^*(Q)$ are Morita equivalent.*

Proof. Let $\widehat{P}$ be the orthogonal frame bundle of $\widehat{Q}$. The following diagram

$$\begin{array}{ccc} \widehat{P} & \xrightarrow{\mathrm{SO}(q)} & \widehat{Q} \\ \Gamma\big\downarrow & & \big\downarrow\Gamma \\ P & \xrightarrow[\mathrm{SO}(q)]{} & Q \end{array}$$

commutes. Since Γ and $\mathrm{SO}(q)$ act freely on $\widehat{P}$ and their actions commute, by a theorem of P. Green (see [15]), $C^*(\widehat{P}/\Gamma, \mathrm{SO}(q))$ is Morita equivalent to $C^*(\widehat{P}/\mathrm{SO}(q), \Gamma)$. □

Hence an elliptic pseudodifferential elliptic operator D on Q determines a class

$$[\widehat{D}] \in KK(C_0(\widehat{Q}) \rtimes \Gamma, \mathbf{C}) \cong KK_\Gamma(C_0(\widehat{Q}), \mathbf{C}).$$

To define the K-theoretical Γ-index $\mathrm{IND}_a(\widehat{D})$ of D, which is an element of $K_0(C^*(\Gamma))$, we first observe that since Γ acts on $L^2(\hat{\eta}^{(i)})$, so also $C^*(\Gamma)$ does, in a canonical way. Now $\widehat{D}$ is a Fredholm operator between $L^2(\eta^{(0)})$ and $L^2(\eta^{(1)})$, and so it determines an element of $K_0(C^*(\Gamma))$, which we call $\mathrm{IND}_a(\widehat{D})$ (c.f. [10, §4]). $\mathrm{IND}_a(\widehat{D})$ is represented by the projections of $L^2(\eta^{(0)})$ onto $\operatorname{Ker}(\widehat{D})$ and of $L^2(\eta^{(1)})$ onto $\operatorname{Ker}(\widehat{D}^*)$ The following theorem can be recovered from a theorem of Kasparov [9].

Theorem 3. *Let D be a pseudodifferential elliptic operator on the good orbifold Q, $D\colon L^2(\eta^{(0)}) \to L^2(\eta^{(1)})$. Let $\widehat{Q}$ be the universal cover of Q, $\Gamma = \pi_1^{\mathrm{ORB}}(Q)$, and $\widehat{D}$ be the operator D lifted to $\widehat{Q}$. Then,*

$$\mathrm{IND}_a(\widehat{D}) = \mathrm{IND}_t(\widehat{D}) \quad \text{in } KK(\mathbf{C}, C^*(\Gamma)),$$

with

$$\mathrm{IND}_t(\widehat{D}) \underset{\text{def}}{=} [C] \bigotimes_{C_0(\widehat{Q}) \rtimes \Gamma} j_\Gamma([\widehat{D}]),$$

where $[\widehat{D}] \in KK_\Gamma(C_0(\widehat{Q}), \mathbf{C})$, $j_\Gamma\colon KK_\Gamma(A, B) \to KK(A \rtimes \Gamma, B \rtimes \Gamma)$ is the canonical homomorphism, and $[C] \in K_0(C_0(\widehat{Q}) \rtimes \Gamma)$ is determined by the projection $p(x, g) = \sqrt{c(x)c(g^{-1}x)}$, where $c \in C_c^\infty(\widehat{Q})$ is such that $\int_\Gamma c(xg)\,dg = 1$ and where $c \geq 0$.

Note that we could also have an index theorem with coefficients in a C^*-bundle rather than in a vector bundle in the spirit of [14].

As a corollary to Theorem 3 we obtain.

Theorem 4. *Let D, Q, $\widehat{Q}$, Γ, and $\widehat{D}$ be as in Theorem 3. Then,*

$$\tau(\mathrm{IND}_a(\widehat{D})) = \text{AS-ind}(D),$$

where τ is the canonical trace on $K_0(C^*(\Gamma)) = (\text{Idempotents of } C^*(\Gamma)\otimes\mathscr{K})/\sim$.

Proof. τ is given by $\tau(a \otimes A) = \tau^\Gamma(a) \otimes T(A)$ where $a \in C^*(\Gamma)$, $a = \sum_{g\in\Gamma}\lambda_g[g]$, $\lambda_g \in \mathbf{R}$, $\tau^\Gamma(a) = \lambda_e$, $e =$ unit of Γ, $A \in \mathscr{K} = \mathscr{K}(L^2(Q))$, $T=$ canonical trace on $\mathscr{K}(L^2(Q)) \subseteq \mathscr{B}(L^2(Q))$. This trace coincides with the trace in [2, p. 57]. □

3. Relations between indices

As we have seen in §1 and in §2, if Q is a good orbifold and D is a pseudodifferential elliptic operator on Q, then we can define the two indices $\mathrm{Ind}_a(D)$ and AS-ind(D). The first one is necessarily an integer, while the second one is a rational number. In general they do not coincide, but there is an interesting relation between them, which is a corollary of the main theorems in [13] and [2] (c.f. also [12, III]). In fact Atiyah's argument applies also to the case in where the action is not free.

Theorem 5. *Let D, Q, $\widehat{Q}$, Γ, and $\widehat{D}$ be as in Theorem 3. Then,*

$$\mathrm{Ind}_a(D) = \text{AS-ind}(D) + R,$$

where (with the notation as in the introduction in [13]),

$$R = \sum_{i=1}^{c} \int \frac{1}{m_i}((-1)^{|\Sigma_i|}\langle ch^\Sigma(D)\mathscr{T}^\Sigma(Q), [\Sigma_i]\rangle),$$

with Σ_i running over the strata of Q.

For example, if $\mathscr{E}$ is the Euler operator on Q, then $\mathrm{Ind}_a(\mathscr{E})$ is equal to the Euler characteristic of Q as a vector space (c.f. [11, PROPOSITION]) and AS-ind$(\mathscr{E}) = X_S(Q)$, where $X_S(Q)$ is the Euler-Satake characteristic of Q, by the Gauss-Bonnet theorem for orbifolds of Satake [16] and the general formula in [12]. Since R depends only on the singular structure of Q, it follows that R is 0 if Q is a smooth manifold, and so in that case we recover the main theorem in [2] from Theorem 5, $\mathrm{Ind}_a(D) = \text{AS-ind}(D)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, CAMPUS BOX 426, BOULDER, COLORADO, 80309

E-mail address: farsi@euclid.colorado.edu