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TODA FLOWS AND ISOSPECTRAL MANIFOLDS

LEONID FAYBUSOVICH

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ABSTRACT. We apply Bott's method to the calculation of Betti numbers of isospectral manifolds. Necessary properties of Toda flows, including a description of the phase portrait, are given with complete proofs.

1. INTRODUCTION

In the present note we show that Bott's method for the calculation of the Betti numbers of adjoint orbits of compact Lie groups [2, 3] can be naturally extended to the case of more general isospectral manifolds. These manifolds appear as a generalization of the manifold of tridiagonal symmetric matrices with a fixed spectrum [4, 8]. We use generalized Toda flows [12–14] (following mainly [8]) as an important technical tool. In particular, we describe complete phase portraits of generalized Toda flows and, as a consequence, show that these dynamical systems (and their discrete analogues, so-called QR-algorithms) possess the Morse-Smale property, i.e., are structurally stable (see in this respect also [4]). The obtained formulas for Betti numbers of isospectral manifolds are direct generalizations of those for adjoint orbits of compact Lie groups [3].

2. PHASE PORTRAIT OF TODA FLOWS

Let $(L, [\ ,\ ])$ be a real Lie algebra of a semisimple noncompact Lie group G with a finite center. Throughout this paper we fix a Cartan involution $\theta: L \to L$ and the corresponding Cartan decomposition

$$L = K \oplus P,$$

where K is a Lie algebra of a maximal compact subgroup U of G. We denote as $\langle\ ,\ \rangle$ the Killing form of L and as $(\ ,\ )$ the corresponding (positive definite) scalar product

$$(x, y) = -\langle x, \theta y\rangle . \tag{1}$$

We also fix a maximal (commutative) subalgebra $A \subset P$ and the corresponding decomposition

$$L = L_0 \oplus \sum_{\alpha\in\Delta} L_\alpha \tag{2}$$

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where Δ is a restricted root system on A (see [11] in connection with necessary Lie-algebraic notions). Let C be a Weyl chamber in A and Δ^+ (Δ^-) be the set of positive (negative) roots relative to this chamber. Consider the Iwasawa decomposition of L

$$L = K \oplus A \oplus \left(\sum_{\alpha \in \Delta^+} L_\alpha \right) . \tag{3}$$

Let $\pi: L \to L$ be the projection of L onto K along $A \oplus \sum\{L_\alpha : \alpha \in \Delta^+\}$. A dynamical system

$$x = [x , \pi(x)] , \qquad x \in L , \tag{4}$$

is called a generalized Toda flow. We denote as Δ_s^+ the set of simple positive roots relative to the Weyl chamber C. Given $\alpha \in \Delta$,

$$\alpha = \sum_{\beta \in \Delta_s^+} n_\beta \beta , \qquad n_\beta \in \mathbb{Z} ,$$

we denote as $\mathrm{h}(\alpha)$ (height of α) the integer

$$\mathrm{h}(\alpha) = \sum_{\beta \in \Delta_s^+} n_\beta .$$

Let

$$V_k = \left(\sum \{L_\alpha : \alpha \in \Delta,\ \mathrm{h}(\alpha) \geq -\kappa\} \right) + L_0 , \qquad W_k = V_k \cap P . \tag{5}$$

Proposition 1. *For every $k \in \mathbb{N}$ the vector subspace W_k is an invariant manifold for* (4).

Proof. Let $\rho: L \to L$ be the projection onto $A \oplus \sum\{L_\alpha : \alpha \in \Delta^+\}$ along K (see (2)). Clearly $[x , \pi(x)] = [\rho(x) , x]$. Further,

$$A \subset L_0 , \qquad [L_\alpha , L_\beta] \subset L_{\alpha+\beta} , \qquad \alpha , \beta \in \Delta \backslash \{0\} . \tag{6}$$

By virtue of (6) we have

$$\left[\left(A \oplus \sum_{\alpha \in \Delta^+} L_\alpha \right) , V_k \right] \subset V_k , \qquad k \in \mathbb{N} ,$$

i.e., $[x , \pi(x)] = [\rho(x) , x] \in V_k$, provided $x \in V_k$.

If $x \in P$, then $[x , \pi(x)] \in P$. Thus, $W_k = P \cap V_k$ is an invariant manifold for (4).

Remark. If we take $L = \mathrm{sl}(n , \mathbb{R})$, $\theta(x) = -x^t$, $k = 1$, we obtain the Flaschka representation for the Toda lattice.

Lemma 1. *Let $n \in C$. Then for $x \in P$*

$$\langle n , [x , \pi(x)] \rangle \geq 0 ,$$

and $\langle n , [x , \pi(x)] \rangle = 0$ if and only if $x \in A$.

Proof. As is easily seen (see e.g., [11])

$$P = A \oplus \left(\sum_{\alpha \in \Delta^+}^{\oplus} F_\alpha \right) , \tag{7}$$

where $F_\alpha = \{x - \theta x : x \in L_\alpha\}$. Let $x \in P$. We have

$$x = a + \sum_{\alpha \in \Delta^+} (x_\alpha - \theta x_\alpha),$$

for some $a \in A$, $x_\alpha \in L_\alpha$. Thus,

$$\pi(x) = -\sum_{\alpha \in \Delta^+} (x_\alpha + \theta x_\alpha).$$

We have

$$\begin{aligned} [x, \pi(x)] = & -\sum \{[a, (x_\alpha + \theta x_\alpha)] : \alpha \in \Delta^+\} \\ & -\sum \{[x_\alpha - \theta x_\alpha, x_\beta + \theta x_\beta] : \alpha \in \Delta^+, \beta \in \Delta^+\}. \end{aligned}$$

Clearly $\theta L_\alpha = L_{-\alpha}$ and $[a, x_\alpha + \theta x_\alpha] = \langle \alpha, a \rangle (x_\alpha - \theta x_\alpha)$; $[x_\alpha - \theta x_\alpha, x_\beta + \theta x_\beta] = ([x_\alpha, x_\beta] - \theta([x_\alpha, x_\beta])) + ([x_\alpha, \theta x_\beta] - \theta([x_\alpha, \theta x_\beta]))$. Here we made use of the following facts: $\theta([x, y]) = [\theta x, \theta y]$, $x, y \in L$, $\theta^2 = Id_L$ (see e.g., [11]). Now $[x_\alpha, x_\beta] - \theta([x_\alpha, x_\beta]) \in F_{\alpha+\beta}$, $([x_\alpha, \theta x_\beta] - \theta[x_\alpha, \theta x_\beta]) \in F_{\alpha-\beta}$ (see [11]) and consequently

$$\begin{aligned} \langle n, [x_\alpha, x_\beta] - \theta([x_\alpha, x_\beta]) \rangle = 0, \qquad & \alpha, \beta \in \Delta^+, \\ \langle n, ([x_\alpha, \theta x_\beta]) - \theta([x_\alpha, \theta x_\beta]) \rangle = 0 \quad & \text{unless } \alpha = \beta. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \langle n, [x, \pi(x)] \rangle = & -\sum \{\langle \alpha, a \rangle \langle n, x_\alpha - \theta x_\alpha \rangle : \alpha \in \Delta^+\} \\ & -\sum \{\langle n, ([x_\alpha, \theta x_\alpha] - \theta([x_\alpha, \theta x_\alpha])) \rangle : \alpha \in \Delta^+\}. \end{aligned}$$

Further, $\langle n, x_\alpha - \theta x_\alpha \rangle = 0$, $\alpha \in \Delta^+$ and $\langle n, [x_\alpha, \theta x_\alpha] \rangle = \langle [n, x_\alpha], \theta x_\alpha \rangle = \langle \alpha, n \rangle \langle x_\alpha, \theta x_\alpha \rangle = -\langle \alpha, n \rangle (x_\alpha, x_\alpha)$; $\langle n, \theta([x_\alpha, \theta x_\alpha]) \rangle = \langle n, [\theta x_\alpha, x_\alpha] \rangle = \langle [n, \theta x_\alpha], x_\alpha \rangle = -\langle \alpha, n \rangle \langle \theta x_\alpha, x_\alpha \rangle = \langle \alpha, n \rangle (x_\alpha, x_\alpha)$. Finally, we have $\langle n, [x, \pi(x)] \rangle = 2\sum \{\langle \alpha, n \rangle (x_\alpha, x_\alpha) : \alpha \in \Delta^+\}$, i.e., $\langle n, [x, \pi(x)] \rangle \geq 0$ and $\langle n, [x, \pi(x)] \rangle = 0$ if and only if $(x_\alpha, x_\alpha) = 0$, $\alpha \in \Delta^+$. In other words, $(\langle n, [x, \pi(x)] \rangle = 0) \Leftrightarrow (x_\alpha = 0, \alpha \in \Delta^+) \Leftrightarrow x \in A$.

Given $p \in P$, denote as $O(p) \subset P$ the orbit of p under the adjoint action of U on P. Since U is a compact Lie group, every orbit $O(p)$ is a smooth manifold. Denote as $T_q(O(p))$ the tangent space to $O(p)$ at a point $q \in O(p)$. Clearly $T_q(O(p)) = \{[q, \xi] : \xi \in K\}$. This implies

Lemma 2. *Every orbit $O(p)$ is an invariant manifold for the dynamical system* (4).

From Lemmas 1 and 2 it follows that

Corollary 1. *The set of the equilibrium points of the dynamical system* (4) *on $O(p)$ coincides with the set $A \cap O(p)$.*

As is known (see, e.g., [11]), for every $p \in P$ the set $A \cap O(p)$ is nonempty, finite, and coincides with an orbit of the Weyl group W, i.e., $A \cap O(p) = Wq$, where q is a unique element of $\overline{C} \cap O(p)$ (we denote as $\overline{C}$ the closure of the Weyl chamber C).

We need the following simple Lemma.

Lemma 3. *Given* $p \in P$, *let* $O(p) \cap A = \{a_1, \ldots, a_k\}$ *for some* $k \in \mathbb{N}$. *There always exists* $n \in C$ *such that* $\langle a_i, n\rangle \neq \langle a_j, n\rangle$, *provided* $i \neq j$.

Proposition 2. *Let* $x(t)$ *be a solution to* (4) *such that* $x(0) = p$. *Then* $x(t) \to p_\pm$, $t \to \pm\infty$, *for some* $p_\pm \in A \cap O(p)$.

Proof. By Lemma 3 there is $n \in C$ such that $\langle n, a\rangle \neq \langle n, b\rangle$ for every $a, b \in A \cap O(p)$, $a \neq b$. The function $f(x) = \langle x, n\rangle$ is a Lyapunov function for (4) on $O(p)$ by Lemma 1. The result follows now by a standard reasoning.

Remark. Proposition 2 is well known (compare with [10]). But our proof seems to be new. We will use the Lyapunov function $x \to \langle x, n\rangle$ as a main technical tool throughout this paper.

Let M (M^*) be the stabilizer (normalizer) of A in U. Then the Weyl group W of L (relative to θ, A) is isomorphic with M^*/M (see [11]). Let G_A and N be connected Lie subgroups of G with Lie algebras A and $\sum\{L_\alpha : \alpha \in \Delta^+\}$, respectively. We denote as B the Borel subgroup MG_AN. As is known (see [11]) B is a closed subgroup of G. Let G/B to the corresponding coset space and $\mu: G \to G/B$ be a canonical projection. The group G acts from the left on G/B. Given $p \in C$, let $\varphi: O(p) \to G/B$ be the mapping such that $\varphi(\mathrm{Ad}(g)p) = g \cdot o$, where $g \in U$, $o = \mu(B)$. The mapping φ defines a smooth isomorphism of $O(p)$ with G/B ([11]).

Let $w \in M^*/M \approx W$ and $w = m^* \cdot M$ for some $m^* \in M^*$. Then $w \cdot o = m^* \cdot o$ does not depend on the choice of a representative $m^* \in M^*$ because $M \subset B$ and $M \cdot o = o$. Recall that [11] the sets $B \cdot (w \cdot o)$, $w \in W$ are pairwise disjoint and, moreover,

$$G/B = \bigcup_{w \in W} B \cdot (w \cdot o), \tag{8}$$

(Bruhat decomposition of G/B). These sets (which, in fact, are submanifolds of G/B) are called the Bruhat cells of this decomposition. We obtain a Bruhat decomposition of $O(p)$, $p \in C$, with the help of the isomorphism φ:

$$O(p) = \bigcup_{w \in W} \varphi^{-1}(B \cdot w \cdot o). \tag{9}$$

Proposition 3. *The Bruhat cells* (9) *are invariant manifolds for the dynamical system* (4).

Proof. Let $x_0 \in O(p)$, $p \in C$ and $\exp(x_0 t) = Q(t)R(t)$, where $t \in \mathbb{R}$, $Q(t) \in U$, $R(t) \in G_AN$. There is such a unique decomposition (Iwasawa decomposition, see [11]). Then $x(t) = \mathrm{Ad}(Q(t)^{-1}) \cdot x_0$ is the solution to (4) such that $x(o) = x_0$ (for a proof see [10]). There is $Q_0 \in U$ such that $\mathrm{Ad}(Q_0^{-1}) \cdot p = x_0$. We have $\varphi(x(t)) = Q(t)^{-1} \cdot Q_0^{-1} \cdot o = R(t) \cdot \exp(-x_0 t) \cdot Q_0^{-1} \cdot o = R(t) \cdot Q_0^{-1} \cdot \exp(-t\,\mathrm{Ad}(Q_0)x_0) \cdot o = R(t) \cdot Q_0^{-1} \cdot \exp(-pt) \cdot o = R(t) \cdot Q_0^{-1} \cdot o = R(t) \cdot \varphi(x_0)$. Here we used that $\exp(-pt) \cdot o = o$, $p \in A$. Now $R(t) \in G_AN \subset B$. Whence $\varphi(x(t))$ belongs to the B-orbit of $\varphi(x_0)$.

Let $w \in W \approx M^*/M$, $w = m^* \cdot M$ for some $m^* \in M^*$. Then the Lie subgroup $m^* \cdot B \cdot (m^*)^{-1}$ does not depend on the choice of a representative $m^* \in M^*$ because M normalizes B. Thus, we obtain the Lie subgroup

$B_w = m^* \cdot B \cdot (m^*)^{-1}$. Given $w_1 \in W$, we obtain from (8) the following decomposition

$$G/B = \bigcup_{w \in W} B_{w_1} \cdot (w \cdot o).$$

There is a unique element $w_m \in W$ such that $w_m \cdot \Delta^+ = \Delta^-$ (see, e.g., [11]). In particular, the following Bruhat decomposition occurs:

$$G/B = \bigcup_{w \in W} B_{w_m} \cdot (w \cdot o). \tag{10}$$

On the other hand, let Θ be the automorphism of G with $T_e(\Theta) = \theta$. Clearly $\Theta \cdot B = B_u$, $u = w_m$. If $\exp(x_0 t) = Q(t) \cdot R(t)$, $Q(t) \in U$, $R(t) \in G_A N$, $x_0 \in P$, and $t \in \mathbb{R}$, then $\exp(\theta(x_0)t) = Q(t)[\Theta R(t)]$, or $Q(t)^{-1} = [\Theta R(t)] \cdot \exp(x_0 t)$. As in the case of proposition 3, we have that $\varphi(x(t))$ belongs to the B_u-orbit of $\varphi(x_0)$, where $u = w_m$. Thus, we proved:

Proposition 4. *The Bruhat cells of the decomposition* (10) (*i.e., B_u-orbits in G/B, $u = w_m$*) *are invariant manifolds for* (4).

Given $n \in C$, consider a mapping $F: \mathbb{R} \times G/B \to G/B$, $F(t, z) = \exp(tn) \cdot z$. Clearly, $F(t_2, F(t_1, z)) = F(t_2 + t_1, z)$, $t_1, t_2 \in \mathbb{R}$. Thus, there exists a vector field X_n on G/B such that $X_n(z) = T_0(F_z) \cdot 1$, $F_z(t) = \exp(tn) \cdot z$, $t \in \mathbb{R}$ and $z \in G/B$. If $w \in W$, $w = m^* \cdot M$, $m^* \in M^*$, then $\exp(tn) \cdot (w \cdot o) = m^* \exp(t \operatorname{Ad}(m^*)^{-1} \cdot n) \cdot o$. But $\operatorname{Ad}(m^*)^{-1} \cdot n \in A$. Whence, $\exp(tn) \cdot (w \cdot o) = w \cdot o$.

Proposition 5. *The Bruhat cell $B \cdot (w \cdot o)$ (respectively, $B_u \cdot (w \cdot o)$, $u = w_m$) is the unstable (respectively, stable) manifold of the equilibrium point $w \cdot o$ of the vector field X_n.*

Proof. We have $B = M \cdot G_A \cdot N$ and every element $y \in N$ has the form $y = \exp(\sum\{x_\alpha : \alpha \in \Delta^+\})$, $a_\alpha \in L_\alpha$ (see [11]). Let $b = c \cdot y$, where $c \in MG_A$. We have

$$\begin{aligned} &\exp(nt) \cdot b \cdot (w \cdot o) \\ &\quad = \exp(nt) \cdot c \cdot \exp(-nt) \cdot \exp\left(\sum\{\exp \operatorname{ad}(nt) \cdot x_\alpha : \alpha \in \Delta^+\}\right) \cdot \exp(nt) \cdot w \cdot o. \end{aligned}$$

But $\exp(nt) \cdot c \cdot \exp(-nt) = c$ because G_A is a commutative group, $\exp(nt) \in G_A$ and M stabilizes G_A. Further, $\operatorname{ad}(nt)x_\alpha = t\langle\alpha, n\rangle x_\alpha$ and consequently $\exp(t \operatorname{ad} n) \cdot x_\alpha = \exp(t\langle\alpha, n\rangle) \cdot x_\alpha \to 0$, $t \to -\infty$ because $\langle\alpha, n\rangle > 0$, $\alpha \in \Delta^+$. This implies $\exp(nt) \cdot b \cdot w \cdot o \to w \cdot o$. Similarly $\exp(nt) \cdot b_m \cdot w \cdot o \to w \cdot o$, $t \to +\infty$, where $b_m \in B_u$, $u = w_m$.

The isomorphism $\varphi: O(p) \to G/B$ enables us to carry over the vector field X_n to the $O(p)$. We denote this vector field as Y_n.

Proposition 6. *Given $n \in C$, the following holds*:

$$\langle n, Y_n(x)\rangle \geq 0, \quad \textit{for every } x \in O(p).$$

Proof. Let $x \in O(p)$ and $\varphi(x) = Q \cdot o$, $Q \in U$. Consider the Iwasawa decomposition

$$\exp(tn) \cdot Q = Q(t) \cdot R(t), \tag{11}$$

where $t \in \mathbb{R}$, $Q(t) \in U$, $R(t) \in G_A N$. The integral curve $x(t)$ of the vector field Y_n such that $x(0) = x$ takes the form: $x(t) = \varphi^{-1}(\exp(nt) \cdot \varphi(x)) = \operatorname{Ad} Q(t) \cdot p$. Differentiating (11) with respect to t, we obtain $\operatorname{Ad}(Q(t)^{-1}) \cdot n = Q(t)^{-1} \cdot dQ(t)/dt + dR(t)/dt \cdot R(t)^{-1}$. Thus, we have

$$Q(t)^{-1} \cdot dQ(t)/dt = \pi(\operatorname{Ad}(Q(t)^{-1}) \cdot n).$$

This implies that

$$\begin{aligned} d/dt(\langle n, x(t)\rangle) &= d/dt(\langle n, \operatorname{Ad}(Q(t)) \cdot p\rangle) = d/dt(\langle \operatorname{Ad}(Q(t)^{-1}) \cdot n, p\rangle) \\ &= (\langle [\operatorname{Ad}(Q(t)^{-1}) \cdot n, \pi(\operatorname{Ad}(Q(t)^{-1}) \cdot n], p\rangle). \end{aligned}$$

This yields

$$\langle n, Y_n(x)\rangle = \langle [\operatorname{Ad}(Q^{-1}) \cdot n, \pi(\operatorname{Ad}(Q^{-1})) \cdot n], p\rangle \geq 0$$

by Lemma 1.

Proposition 7. *Every equilibrium point $w \cdot p$, $w \in W$, of the dynamical system* (4) *on the orbit $O(p)$, $p \in C$, is hyperbolic and*

$$\operatorname{ind}(w \cdot p) = \sum \{\dim L_\alpha : \alpha \in \Delta^+, w^{-1}(\alpha) \in \Delta^+\}.$$

Proof. Denote as $A_w \colon T_{wp}(O(p)) \to T_{wp}(O(p))$ the linearization of the dynamical system (4) at the equilibrium point $w \cdot p$. Clearly A_w is the restriction of the linear operator $\xi \to [wp, \pi(\xi)] \colon P \to P$ to T_{wp}. We have $T_{wp}(O(p)) = \sum_{\alpha \in \Delta^+} F_\alpha$. Let $(\xi_\alpha - \theta\xi_\alpha) \in F_\alpha$, $\xi_\alpha \in L_\alpha$. Then $A_w(\xi_\alpha - \theta\xi_\alpha) = [wp, \pi(\xi_\alpha - \theta\xi_\alpha)] = -[wp, \xi_\alpha + \theta\xi_\alpha] = -\langle \alpha, wp\rangle(\xi_\alpha - \theta\xi_\alpha)$. Thus, $\operatorname{ind} A_w = \sum\{\dim L_\alpha \colon \alpha \in \Delta^+, w^{-1}(\alpha) \in \Delta^+\}$.

Theorem 1. *The Bruhat cell $\varphi(B \cdot w \cdot o)$ (respectively, $\varphi(B_u \cdot w \cdot o)$, $u = w_m$) is the unstable (respectively, stable) manifold for the equilibrium point $w \cdot p$ of the dynamical system* (4) *on the orbit $O(p)$, $p \in C$.*

Remark. For the notations used see (10).

Proof. By Lemma 3 there is $n \in C$ such that $\langle n, w_2 \cdot p\rangle \neq \langle n, w_1 \cdot p\rangle$ if $w_1 \neq w_2$. Denote as f_n the function $x \to \langle n, x\rangle \colon P \to \mathbb{R}$. Let $\varepsilon > 0$ be such that $\varepsilon < \min\{|f_n(w_2 p) - f_n(w_1 p)| \colon w_2 \neq w_1 \in W\}$. Let us choose an open neighbourhood $V_\varepsilon(wp)$ of the point wp in P such that $|f_n(x) - f_n(wp)| < \varepsilon$, $x \in V_\varepsilon(wp)$. Denote as $C(w)$ the cell $\varphi(B_u \cdot w \cdot o)$, $u = w_m$, and as $F \colon \mathbb{R} \times O(p) \to O(p)$, $F_n \colon \mathbb{R} \times O(p) \to O(p)$ the flows of the vector fields (4) and Y_n, respectively. Let $x \in C(w) \cap V_\varepsilon(wp)$. By Proposition 2 $\lim F(t, x) = w_1 \cdot p$, $t \to +\infty$, for some $w_1 \in W$. By Lemma 1, $f_n(w_1 p) \geq f_n(x)$. But $f_n(x) > f_n(wp) - \varepsilon$ because $x \in V_\varepsilon(wp)$. Thus, $f_n(w_1 \cdot p) > f_n(wp) - \varepsilon$. This implies (due to the choice of ε) that $f_n(w_1 \cdot p) \geq f_n(wp)$. Clearly there is $T > 0$ such that $|f_n(F(t, x)) - f_n(w_1 \cdot p)| < \varepsilon$ if $t \geq T$. By Proposition 4, $F(t, x) \in C(w)$, $t \in \mathbb{R}$ and consequently (Proposition 5) $\lim F_n(t, F(t^*, x)) = wp$, $t \to +\infty$ for every $t^* \in \mathbb{R}$. By Proposition 6, $f_n(wp) \geq f_n(F(t^*, x))$ for every $t^* \in \mathbb{R}$. In particular, $f_n(wp) \geq f_n(F(T, x)) > f_n(w_1 \cdot p) - \varepsilon$, i.e., $f_n(wp) \geq f_n(w_1 \cdot p)$. Thus, $f_n(wp) = f_n(w_1 \cdot p)$ and hence $wp = w_1 \cdot p$ by the choice of n. As is known $B_u \cdot w \cdot o$ is a smooth submanifold of G/B and $\dim(B_u \cdot w \cdot o) = \sum\{\dim L_\alpha : \alpha > 0, w^{-1}(\alpha) > 0\}$ (see, e.g., [11]). Here $u = w_n$. Thus, $\dim C(w) = \operatorname{ind}(wp)$ by Proposition 7. If we denote as

$S(wp)$ the stable manifold of (4) on the $O(p)$, then, what we have proven means that there is a neighbourhood $V'(wp)$ of the point wp in P such that $V'(wp) \cap S(wp) = V'(wp) \cap C(w)$. This holds for every $w \in W$. If $x \in C(w)$ and $F(t, x) \to w_1 \cdot p$, $t \to +\infty$, then for sufficiently large t, $F(t, x) \in V'(w_1 \cdot p) \cap S(w_1 \cdot p)$ and hence $F(t, x) \in C(w_1)$. But by virtue of Proposition 4, $F(t, x) \in C(w)$. Thus, $C(w_1) \cap C(w) \neq \varnothing$, i.e., $C(w_1) = C(w)$ and $w_1 = w$. This proves that $S(wp) = C(w)$, $w \in W$. The proof for the case of unstable manifolds is similar (but relies upon Proposition 3 instead of Proposition 4).

Remark. The idea of this proof is due to Atiyah and Bott [1, Proposition 1.19].

Corollary. *The Toda flow* (4) *on the orbit* $O(p)$, $p \in C$, *is a Morse-Smale vector field.*

Proof. We must show that stable and unstable manifolds of (4) intersect each other transversally. Let $x \in S(wp) \cap U(w_1 \cdot p)$, where we denote as $U(wp)$ the unstable manifold of wp, $w \in W$. Then $S(wp) = B_u \cdot x$, $u = w_m$ and $U(w_1 \cdot p) = B \cdot x$ by Theorem 1. But

$$L(B) \oplus L(B_{w_m}) = A \oplus M \oplus \left(\sum_{\alpha>0} L_\alpha\right) \oplus \left(\sum_{\alpha<0} L_\alpha\right) = L.$$

Thus, $T_x(S(wp)) \oplus T_x(U(w_1 \cdot p)) = T_x(O(p))$.

Remark. The so-called QR-algorithm associated with (4) can be described as a discrete dynamical system on $\Pi = \exp(P)$ as follows. If $F: \mathbb{R} \times P \to P$ is the flow of (4) and $G: \mathbb{Z} \times \Pi \to \Pi$ is the flow of the associated QR-algorithm, then

$$G(k, \exp x) = \exp(F(k, x)), \qquad k \in \mathbb{Z},\ x \in P,$$

(see, e.g., [4, 10]). This implies that QR-algorithm, associated with (4) is a Morse-Smale dynamical system on the every orbit $Or(p) = \exp[O(p)]$. This is clear because exp maps P diffeomorphically onto Π.

Remark. Theorem 1 along with Corollary for the case $L = \mathrm{sl}(n, \mathbb{R})$ was proved in [6] with the help of a different technique [5, 7].

3. Isospectral manifolds

The mapping $\psi: U/M \times C \to P$, $\psi(g, p) = \mathrm{Ad}(g)p$ is a diffeomorphism onto an open subset P' in P (see, e.g., [11]). Thus, the mapping $\nu: P' \to C$, $\nu(p) = \pi_2 \cdot \psi^{-1}(p)$, $\pi_2: U \times C \to C$, $\pi_2(g, p) = p$ is a submersion. Clearly $O(p) = \nu^{-1}(\nu(p))$. The set $W_k' = W_k \cap P'$ is an open submanifold of W_k, $k \geq 1$. Denote as ν_k the restriction of ν on W_k'.

Proposition 8. *Given* $p \in C$, *the mapping* $T_x(\nu_k): T_x(W_k') \to T_x(C)$ *is a submersion at every point* $x \in O(p) \cap W_k' = \nu_k^{-1}(\nu_k(p)) = I_k$. *The set* I_k *is a smooth compact orientable manifold*; $\dim I_k = \dim W_k - \dim A$, $k \geq 1$.

Proof. Let $A' = \{a \in A : \langle \alpha, a \rangle \neq 0,\ \forall \alpha \in \Delta\}$. Consider the mapping $i_k: A' \to W_k'$, $i_k(a) = a$. As is known, $A' = \bigcup\{w \cdot C : w \in W\}$ (see, e.g., [11]). If $a \in w \cdot C$, then $\nu_k \cdot i_k(a) = w^{-1}a$. This implies that $T_a(\nu_k \cdot i_k) = T_{ik(a)}(\nu_k) \cdot T_a(i_k): A \to A$ coincides with the mapping $b \to w^{-1}b$, $b \in A$. Consequently $\mathrm{Im}\, T_a(\nu_k) = A$, i.e., ν_k is a submersion at every point $a \in A'$.

Let $F: \mathbb{R} \times P \to P$ be the flow of the dynamical system (4) and $x \in O(p) \cap W_k'$. By Proposition 2, $F(x, t) \to wp$, $t \to +\infty$, for some $w \in W$. Denote as $F_t: P \to P$ the mapping $F_t(y) = F(t, y)$, $t \in \mathbb{R}$. By Proposition 1, $T_y(F_t)$ maps $T_y(W_k) = W_k$ isomorphically onto $T_{F(t,y)}(W_k) = W_k$. By Lemma 2, $\nu_\kappa(F_t(x)) = \nu_k(x)$. Hence

$$T_{F_t(y)}(\nu_k) \circ T_x(F_t) = T_x(\nu_k), \qquad x \in O(p). \tag{11}$$

For sufficiently large t, $T_{F(t,x)}(\nu_k)$ is a surjection because $T_{wp}(\nu_k)$ is a surjection and $F_t(x) \to wp$, $t \to +\infty$. By (11) we obtain that $T_x(\nu_k)$ is also a surjection. Thus, we have proven that ν_k is a submersion at every point $x \in I_k$. This implies that $I_k = \nu_k^{-1}(\nu_k(p))$ is an orientable manifold; $\dim I_k = \dim W_k - \dim A$.

Let us observe that $O(p)$ and W_k intersect each other transversally at the points wp, $w \in W$. In particular $T_{wp}(I_k) = T_{wp}(O(p)) \cap W_k$, i.e.,

$$T_{wp}(I_k) = \sum_{\substack{\alpha \in \Delta^+ \\ h(\alpha) \leq k}} F_\alpha. \tag{12}$$

Remark. The idea of this proof is due to Fried [8], who considered the case $L = \mathrm{sl}(n, \mathbb{R})$.

Corollary. *For every $k \geq 1$, I_k is a connected manifold.*

Proof. Let Y_k be the restriction of the Toda vector field (4) to I_k. As in the proof of Proposition 7 (taking into account (12)) we show that the index of Y_k at wp, which we denote as $\mathrm{ind}_k(wp)$ takes the form

$$\mathrm{ind}_k(wp) = \sum \{\dim L_\alpha;\ \alpha \in \Delta^+,\ w^{-1}(\alpha) \in \Delta^+,\ h(\alpha) \leq k\}.$$

In particular, there is exactly one stable equilibrium point p and $\mathrm{ind}_k(p) = \dim I_k$. Indeed, if $w \neq e$, then there is $\alpha \in \Delta_s^+$ (a simple positive root relative to C) such that $w^{-1}(\alpha) < 0$ (see, e.g., [11]). But $(\alpha \in \Delta_s^+) \Rightarrow (h(\alpha) = 1 \leq k)$. Clearly the stable manifold of p on I_k is an open, dense, and connected set in I_k. In particular, I_k is also a connected set.

Let $p \in C$ and f be a smooth function, defined on some open neighbourhood of the orbit $O(p)$. Consider the problem

$$f(q) \to \min, \qquad q \in O(p). \tag{13}$$

Proposition 9. *The stationary points of the problem* (13) *are exactly the points $q_0 \in O(p)$ such that*

$$[\nabla f(q_0), q_0] = 0. \tag{14}$$

Here $\nabla f(q) \in P$ is by definition such that $\langle \nabla f(q), \xi \rangle = df(q) \cdot \xi$, $\xi \in P$. If q_0 is a stationary point of (13), *then the Hessian $H(q_0)$ at this point takes the form*

$$\begin{aligned} H(q_0)([\xi, q_0], [\xi, q_0]) &= D^2 f(q_0) \cdot ([\xi, q_0], [\xi, q_0]) \\ &\quad + \langle \nabla f(q_0), [\xi, [\xi, q_0]] \rangle, \qquad \xi \in K. \end{aligned} \tag{15}$$

Proof. Clearly $T_q(O(p)) = [\xi, q]$, $\xi \in K$. Thus, $(df(q)([\xi, q]) = 0,\ \forall \xi \in K) \Leftrightarrow (\langle \nabla f(q), [\xi, q] \rangle = 0,\ \forall \xi \in K) \Leftrightarrow ([\nabla f(q), q] = 0)$. If $\mu: N_1 \to N_2$ is a submersion of manifolds N_i, $i = 1, 2$ and $r: N_2 \to \mathbb{R}$ is a smooth function, then, given a critical point z of r, every $z' \in \mu^{-1}(z)$ is a critical point of $r \cdot \mu$. Moreover, if H_z and $H_{z'}$ are Hessians of r and $r \cdot \mu$ at the points z and z', respectively, then $H_z(T_{z'}(\mu)\cdot\xi, T_{z'}(\mu)\cdot\xi) = H_{z'}(\xi, \xi)$, $\xi \in T_{z'}(N_2)$. This is verified by a direct calculation in local coordinates.

If q_0 is a critical point of f, then the mapping $\xi \to \exp(\operatorname{ad}\xi)\cdot q_0$ is a submersion of a neighbourhood of zero in K onto some neighbourhood of q_0 in $O(p)$. Consider the function $\varphi(\xi) = f(\exp(\operatorname{ad}\xi)\cdot q_0)$, $\xi \in K$. We have $\varphi(\xi) = f(q_0 + [\xi, q_0] + 1/2([\xi, [\xi, q_0]]) + o(\|\xi\|^2)) = f(q_0) + Df(q_0)\cdot([\xi, q_0] + 1/2([\xi, [\xi, q_0]])) + 1/2(D^2 f(q_0)([\xi, q_0], [\xi, q_0])) + o(\|\xi\|^2)$. Hence $D^2\varphi(0)(\xi, \xi) = Df(q_0)\cdot[\xi, [\xi, q_0]] + D^2 f(q_0)([\xi, q_0], [\xi, q_0])$. But $H(q_0)([\xi, q_0], [\xi, q_0]) = D^2\varphi(0)(\xi, \xi)$.

Consider the special case of the problem (13): $f(q) = f_n(q) = \langle n, q\rangle$, $n \in C$. By Proposition 9 the stationary points are determined by the condition $[q, n] = 0$, $q \in O(p)$, i.e., the set of stationary points coincides with $A \cap O(p) = W\cdot p$. Our ultimate goal is the following problem

$$\langle n, q\rangle \to \min, \qquad q \in (O(p) \cap W_k) = I_k, \qquad k \geq 1. \tag{16}$$

Proposition 10. *The set of stationary points for the problem* (16) *is* $A \cap O(p) = W\cdot p$. *The Hessian* $H_k(wp)$ *at the point* wp, $w \in W$ *takes the form*

$$H_k(wp)([\xi, wp], [\xi, wp]) = -\sum_{\substack{\alpha>0\\ h(\alpha)\leq k}} \langle w^{-1}\alpha, p\rangle\langle\alpha, n\rangle(\xi_\alpha - \theta\xi_\alpha, \xi_\alpha - \theta\xi_\alpha), \tag{17}$$

where $\xi = \sum\{(\xi_\alpha + \theta\xi_\alpha) : \alpha > 0,\ h(\alpha) \leq k\}$, $\xi_\alpha \in L_\alpha$. *In particular, every critical point is nondegenerate and*

$$\operatorname{ind} H_k(wp) = \sum_{\substack{\alpha>0\\ h(\alpha)\leq k\\ w^{-1}(\alpha)>0}} (\dim L_\alpha). \tag{18}$$

Proof. Since I_k is a submanifold of $O(P)$, every point from $A \cap O(p)$ is also a stationary point of I_k. Let $x \in I_\kappa$, $x \notin A \cap O(p)$. Then by Proposition 1 and Lemma 2, $[x, \pi(x)] \in T_x(I_k)$. But $df(x)([x, \pi(x)]) = \langle n, [x, \pi(x)]\rangle > 0$ by Lemma 1. Hence, x is not a stationary point for f on I_k. The explicit expression for the Hessian $H_k(wp)$ follows from (12) and (15) by a direct calculation.

Let L be the Lie algebra of a complex semisimple Lie group. We consider L as a Lie algebra over $\mathbb{R}$. But L is endowed with a complex structure. Let $(h_\alpha, e_\alpha, e_{-\alpha})$, $\alpha \in \Delta^+$, be a Chevalley basis for L, i.e., $h_\alpha = [e_\alpha, e_{-\alpha}]$, $\alpha \in \Delta^+$, $[e_\alpha, e_\beta] = N_{\alpha,\beta}\cdot e_{\alpha+\beta}$, where $N_{\alpha,\beta} \in \mathbb{R}$, $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Delta \cup \{0\}$, $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$, $\alpha, \beta \in \Delta$. Let, further, $H = \operatorname{span}_{\mathbb{C}}(h_\alpha, \alpha \in \Delta^+)$, $H_{\mathbb{R}} = \{h \in H : \alpha(h) \in \mathbb{R},\ \forall\alpha \in \Delta\}$. Set $f_\alpha = e_\alpha - e_{-\alpha}$, $g_\alpha = i(e_\alpha + e_{-\alpha})$, $\alpha \in \Delta^+$; $K = \operatorname{span}_{\mathbb{R}}(f_\alpha, g_\alpha, \alpha \in \Delta^+) \oplus iH_{\mathbb{R}}$, $A = H_{\mathbb{R}}$, $L_\alpha = \mathbb{C}e_\alpha$, $\alpha \in \Delta$, $P = \operatorname{span}_{\mathbb{R}}(if_\alpha, ig_\alpha) \oplus H_{\mathbb{R}}$. Clearly $L = K \oplus P$ is a Cartan decomposition and $\dim_{\mathbb{R}} L_\alpha = 2$, $\alpha \in \Delta$.

Theorem 2. *In the situation described above* (*i.e., for the case that* L *is the Lie algebra of a complex semisimple Lie group*) *the Betti numbers* $\beta_s(I_k)$, $k \geq 1$, $0 \leq s \leq \dim I_k$ *take the form*

$$\beta_{2l+1}(I_k) = 0, \qquad \beta_{2l}(I_k) = \operatorname{card}\{w \in W : \gamma_\kappa(w) = l\}, \tag{19}$$

where $\gamma_k(w) = \text{card}\{\alpha > 0 : w^{-1}(\alpha) > 0,\ h(\alpha) \le k\}$. *Besides,* I_k *are simply connected manifolds.*

Proof. By Proposition 10 in this situation all indices $\operatorname{ind} H_k(wp)$ are even. Hence f_n is a perfect Morse function on I_k (see, e.g., [1]). The expression for Betti numbers follows from (18). From the proof of Corollary of Proposition 7 we know that the restriction of the Toda vector field to I_k has exactly one stable equilibrium point whose stable manifold is open, dense in I_k and is clearly simply connected. In the situation considered stable manifolds of other equilibrium points have codimensional greater or equal to 2. Thus, I_k is itself simply connected.

Remark. The Betti numbers (19) coincide with those of a so-called Hessenberg varieties (see [9, 15]). This is not accidental because isospectral manifolds I_k and corresponding Hessenberg varieties coincide modulo action of a toral group.

Remark. If L is the Lie algebra of a complex semisimple Lie group as above, we can consider the real split form L_s of L, $L_s = (\sum_{\alpha\in\Delta}\mathbb{R}e_\alpha)\oplus H_{\mathbb{R}}$. If we take

$$K_s = \sum_{\alpha\in\Delta^+}\mathbb{R}(e_\alpha - e_{-\alpha}), \qquad P_s = H_{\mathbb{R}} \oplus \sum_{\alpha\in\Delta^+}\mathbb{R}(e_\alpha + e_{-\alpha}),$$

$A = H_{\mathbb{R}}$, we obtain a Cartan decomposition for L_s. As it follows from the results of [4, 8], the function f_n on I_1 in this situation is also a perfect Morse function. It seems that the case of a general semisimple Lie algebra over $\mathbb{R}$ can be considered with the help of the technique developed in [1].

4. Concluding remarks

In the present paper we have described a complete phase portrait of a generalized Toda flow on a semisimple noncompact Lie algebra over $\mathbb{R}$ and of a corresponding QR-algorithm. In particular, we have shown that these dynamical systems possess the Morse-Smale property (e.g., structurally stable). We have applied the results obtained to the study of topological properties of isospectral manifolds I_k that are natural generalizations of the isospectral manifold of tridiagonal symmetric matrices. The study of these manifolds is a necessary preliminary step towards a solution of the inverse spectral problem for the situation considered.

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Department of Mathematics, University of Notre Dame, P.O. Box 398, Notre Dame, Indiana 46556

E-mail address: leonid@cartan.math.nd.edu