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AN ORDER THEORETIC CHARACTERIZATION OF PEANO CONTINUA

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ABSTRACT. A characterization of Peano continua as images of dendrites is obtained which allows us to characterize Peano continua order-theoretically.

1. INTRODUCTION

Order theoretic characterizations have been obtained for a number of topological spaces which are arc-connected and acyclic. Spaces which have yielded such characterizations include dendrites and trees [W2], dendroids [W5], dendritic spaces [W6], and topologically chained continua [W3]. However, for spaces which are not acyclic, only local trees have been characterized order theoretically. One such characterization is due to L. E. Ward [W4], and another is due to G. Dimitroff [D]. Furthermore, both Ward [W1] and V. Knight [Kn] have described partial orders for Peano continua. Ward [W1] shows that any Peano continuum may be partially ordered in terms of its cutpoints. Knight [Kn] constructs a continuous partial order for Peano continua, and uses it to show that any Peano continuum is arc-connected. However, neither of these partial orders characterizes Peano continua.

In this paper, an order theoretic characterization of Peano continua is obtained. Peano continua are characterized as images of dendrites in such a way that we are able to partially order them using the partial order on the dendrites described in [W2]. It turns out that this partial order characterizes Peano continua.

2. PRELIMINARIES

A *continuum* is a compact connected Hausdorff space. A locally connected metric continuum is a Peano continuum.

A connected Hausdorff space is *dendritic* if every pair of distinct points in the space can be separated by a third point in the space. A compact, dendritic space is a *tree*. A *dendrite* is a Peano continuum which contains no simple closed

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curves. It is known (Theorem V.1.1 of [Wh]) that a dendrite is a metrizable tree.

The space X is said to be *rim-finite*, or *regular*, if every point has arbitrarily small neighborhoods which have finite boundaries, and X is said to be *rim-compact* if every point has arbitrarily small neighborhoods with compact boundaries. A point $x \in X$ is an *endpoint* of X if there exists arbitrarily small neighborhoods of x with one point boundaries. We will let $E(X)$ denote the set of all endpoints of X.

If X is a continuum, $\leq$ a partial order on X, and $x \in X$, we will use the following notation:

$$L(x) = \{y \in X : y \leq x\} \quad \text{and} \quad M(x) = \{y \in X : x \leq y\}.$$

If $\{(x, y) : x \leq y\}$ is a closed subset of $X \times X$, we say that $\leq$ is a *continuous* partial order. If $L(x)$ and $M(x)$ are closed for each $x \in X$, we say that the partial order is *semicontinuous*, and call X a *partially ordered topological space* (POTS). It is easy to see that every continuous partial order is semicontinuous. A partial order is *order dense* if for each pair of elements, x and y, such that $x < y$, there exists $z \in X$ such that $x < z < y$. We will let $\text{Max}(X)$ denote the set of all maximal elements of X, and $\text{Min}(X)$ denote the set of all minimal elements of X. We say that an element $x \in X$ is a *zero* if $x \leq y$ for all $y \in X$. We conclude this section with a theorem which is originally due to A. D. Wallace [Wa].

Theorem 2.1 (Theorem 1, [W1]). *A compact POTS contains a maximal element.*

3. Two characterizations of Peano continua

The first theorem of this section characterizes Peano continua as images of dendrites under maps which are one-to-one except on the endpoints of the dendrite. This theorem will allow us to partially order any Peano continuum using the cutpoint partial order on the dendrite described in [W2]. This partial order will then be used to characterize Peano continua order theoretically.

Theorem 3.1. *If X is a metric space, then X is a Peano continuum if and only if $X = \phi(D)$ where D is a dendrite and ϕ is a map satisfying the following conditions:*

(i) $\phi|_{D-E(D)}$ *is one-to-one,*
(ii) $\phi(E(D)) \cap \phi(D - E(D)) = \varnothing$.

Before proving the theorem we state two lemmas.

Lemma 3.2 (Lemma 2, [W7]). *If D_1 and D_2 are dendrites with $D_1 \subset D_2$, then there exists a retraction $r : D_2 \to D_1$ which is monotone.*

Lemma 3.3. *If D is a dendrite and Z is a dense connected subset of D, then $D - Z \subset E(D)$.*

Proof. Suppose that there exists $d \in D - Z$ such that d is not an endpoint of D. Then $D - \{d\}$ has at least two components. Let C_1 be the component of $D - \{d\}$ which contains Z and let C_2 be another component of $D - \{d\}$. Then C_2 is open since D is locally connected and $Z \subset D - C_2$. Therefore, $\overline{Z} \subset D - C_2$, a contradiction since $\overline{Z} = D$. Thus $D - Z \subset E(D)$.

We are now ready for the proof of Theorem 3.1. First suppose that $X = \phi(D)$ where D is a dendrite and ϕ is a map. Then X is the image under a closed map of a locally connected continuum, so that X is a locally connected continuum.

Now suppose that X is a Peano continuum. By Lemma 2.3 of [W8], there exists a nested sequence of dendrites $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ satisfying the following conditions:

(i) if $Y = \bigcup_{n=0}^{\infty} D_n$, then $\overline{Y} = X$,
(ii) if C is a component of $D_{n+1} - D_n$, then $\operatorname{diam}(C) < 2^{-n}$,
(iii) if E is a component of $D_{n+i} - D_n$ ($i \geq 1$) whose one point boundary in D_{n+i} is an element of $D_n - D_{n-1}$, and if K is the component of $Y - D_n$ which contains E, then $\overline{K} \cap D_{n-1} = \varnothing$.

For each $D_n \subset D_{n+1}$, let $r_n : D_{n+1} \to D_n$ be the retraction of Lemma 3.2, and let

$$D_\infty = \varprojlim \{D_n, r_n\}$$

where $\varprojlim \{D_n, r_n\}$ is the inverse limit space of the sequence $\{D_n, r_n\}$. Lemma 1 of [W7] shows that D_∞ is a dendrite. Define $\phi : D_\infty \to X$ by

$$\phi(d) = \lim d_n$$

where $d = (d_1, d_2, \ldots)$. Lemma 4 of [W7] shows that ϕ is a well-defined continuous surjection. Let

$$Z = \{d \in D_\infty : d \text{ is eventually constant}\}.$$

It follows from the proof of Theorem 1.1 in [W8] that $\phi|_Z : Z \to Y$ is a bijection, $\phi(D_\infty - Z) \subset X - Y$, and $\overline{Z} = D_\infty$. Furthermore, it is easy to check that $\phi^{-1} : Y \to Z$ is a continuous surjection. Since Y is connected, it follows that Z is connected. Therefore, $Z \subset D_\infty$ with Z connected and $\overline{Z} = D_\infty$. Lemma 3.3 shows that $D_\infty - Z \subset E(D_\infty)$. This implies that $D_\infty - E(D_\infty) \subset Z$ and so $\phi|_{D_\infty - E(D_\infty)}$ is one-to-one. In addition, since $\phi(D_\infty - Z) \subset X - Y$, it follows that $\phi(D_\infty - Z) \cap \phi(Z) = \varnothing$ and so $\phi(E(D)) \cap \phi(D - E(D)) = \varnothing$. This completes the proof of Theorem 3.1.

The next lemma will be useful in the order theoretic characterization of Peano continua.

Lemma 3.4. *Let X be a Peano continuum, $\phi : D \to X$ a map such that $\phi|_{D-E(D)}$ is one-to-one and $\phi(D - E(D)) \cap \phi(E(D)) = \varnothing$. If $A(X) = \{x : |\phi^{-1}(x)| > 1\}$, then $X - A(X)$ is a connected, arc-connected, locally connected, regular, dendritic subset of X.*

Proof. It is easy to check that $X - A(X)$ is homeomorphic to $D - \phi^{-1}(A(X))$. Furthermore, since $\phi|_{D-E(D)}$ is one-to-one and $\phi(D - E(D)) \cap \phi(E(D)) = \varnothing$, it follows that $\phi^{-1}(A(X))$ is a subset of $E(D)$. Thus $D - \phi^{-1}(A(X))$ is connected, arc-connected, locally connected, regular, and dendritic and so $X - A(X)$ has the same properties.

Before using Theorem 3.1 to describe a partial order for Peano continua, it will be useful to recall the partial order for dendrites described in [W2]. It is known as the cutpoint partial order with respect to some basepoint e. Let D be a dendrite, and $e \in D$. Define a relation $\leq_e$ on D as follows: $x \leq_e y$ if and only if $x = e$, or $x = y$, or x separates e and y. We write $x <_e y$ if $x \leq_e y$ and $x \neq y$. It is shown in [W2] that $\leq_e$ satisfies the following conditions:

(i) $\leq_e$ is continuous,
(ii) $\leq_e$ is order dense,
(iii) for $x \in D$, $y \in D$, it follows that $L(x) \cap L(y)$ is a nonnull chain,
(iv) $M(x) - \{x\}$ is an open set for each $x \in D$.

Furthermore, e is zero with respect to this partial order. Finally note that if e is not an endpoint of D then $E(D) = \mathrm{Max}(D)$.

We are now ready to use Theorem 3.1 to describe a partial order for any Peano continuum. Let X be a Peano continuum. By Theorem 3.1, there exists a dendrite D, and a map $\phi : D \to X$ such that $\phi|_{D-E(D)}$ is one-to-one and $\phi(D - E(D)) \cap \phi(E(D)) = \varnothing$. Let e be an element of $D - E(D)$, and let $\leq_e$ be the cutpoint partial order on D with respect to e. Define a binary relation $\leq_D$ on X as follows: $x \leq_D y$ if and only if there exists $p \in \phi^{-1}(x)$ and $q \in \phi^{-1}(y)$ such that $p \leq_e q$. We write $x <_D y$ if $x \leq_D y$ and $x \neq y$. Let $A(X) = \{x \in X : |\phi^{-1}(x)| > 1\}$.

Lemma 3.5. *If $x \in A(X)$ and $y \in X$ such that $x \leq_D y$, then $x = y$.*

Proof. If $x \in A(X)$ it follows that $\phi^{-1}(x) \subset E(D)$. Furthermore, since $x \leq_D y$, there exists $p \in \phi^{-1}(x)$ and $q \in \phi^{-1}(y)$ such that $p \leq_e q$. But p is an element of $E(D) \subset \mathrm{Max}(D)$ and hence $p = q$. Therefore $x = y$ since $x = \phi(p) = \phi(q) = y$.

Lemma 3.6. *If x is an element of X such that $x <_D y$ for some $y \in X$, then $|\phi^{-1}(x)| = 1$.*

Proof. Suppose $|\phi^{-1}(x)| > 1$. Then $x \in A(X)$ which implies that $x = y$ by Lemma 3.5. This is a contradiction so it must be the case that $|\phi^{-1}(x)| = 1$.

Proposition 3.7. *The binary relation $\leq_D$ on X is a continuous, order dense partial order on X.*

Proof. First note that $\leq_D$ is reflexive. To see that $\leq_D$ is antisymmetric, suppose that $x \leq_D y$ and $y \leq_D x$. If $x \neq y$, then $x <_D y$ and $y <_D x$. But $x <_D y$ implies that $|\phi^{-1}(x)| = 1$, and $y <_D x$ implies that $|\phi^{-1}(y)| = 1$. It follows that $\phi^{-1}(x) \leq_e \phi^{-1}(y)$ and $\phi^{-1}(y) \leq_e \phi^{-1}(x)$ which implies that $\phi^{-1}(x) = \phi^{-1}(y)$ since $\leq_e$ is antisymmetric. But this implies that $x = y$, a contradiction. It follows that $\leq_D$ is antisymmetric.

To see that $\leq_D$ is transitive, suppose that $x <_D y$ and $y <_D z$. Lemma 3.6 implies that $|\phi^{-1}(x)| = |\phi^{-1}(y)| = 1$, and that $\phi^{-1}(x) \leq_e \phi^{-1}(y)$. Also, there exists $r \in \phi^{-1}(z)$ such that $\phi^{-1}(y) \leq_e r$. Since $\leq_e$ is transitive, it follows that $\phi^{-1}(x) \leq_e r$, which implies that $x \leq_D z$. Therefore $\leq_D$ is transitive, and it follows that $\leq_D$ is a partial order on X.

In order to see that $\leq_D$ is continuous, we note that

$$\{(x, y) \in X \times X : x \leq_D y\} = (\phi \times \phi)(\{(p, q) \in D \times D : p \leq_e q\}$$

where $(\phi \times \phi)((p, q)) = (\phi(p), \phi(q))$. Now it is easy to see that $\{(x, y) \in X \times X : x \leq_D y\}$ is closed, since $\{(p, q) \in D \times D : p \leq_e q\}$ is closed in $D \times D$, and $(\phi \times \phi) : D \times D \to X \times X$ is a closed map. Therefore $\leq_D$ is a continuous partial order.

Finally, we want to show that $\leq_D$ is order dense, so suppose that $x <_D y$. Then $|\phi^{-1}(x)| = 1$, and there exists $q \in \phi^{-1}(y)$ such that $\phi^{-1}(x) <_e q$. Since

$\leq_e$ is order dense, there exists r such that $\phi^{-1}(x) <_e r <_e q$ which implies that $x <_D \phi(r) <_D y$ so that $\leq_D$ is order dense. This completes the proof of the proposition.

The next three lemmas record some useful properties of the partial order $\leq_D$.

Lemma 3.8. *If $x \in X$, then*

$$M(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(M(p)) \quad \textit{and} \quad L(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(L(p)).$$

Proof. Let $x \in X$. Suppose $y \in M(x)$. Then $x \leq_D y$ which implies that there exist $p \in \phi^{-1}(x)$ and $q \in \phi^{-1}(y)$ such that $p \leq_e q$. It follows that q is an element of $M(p)$ which implies that $y \in \phi(M(p))$ where $p \in \phi^{-1}(x)$. Now suppose that $y \in \bigcup_{p\in\phi^{-1}(x)} \phi(M(p))$. Then $y = \phi(q)$ where $q \in M(p)$ for some p in $\phi^{-1}(x)$. As a consequence $p \leq_e q$, hence $x \leq_D y$. Therefore $y \in M(x)$. Thus we have proved that $M(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(M(p))$. A similar proof shows that $L(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(L(p))$.

Lemma 3.9. *$\phi(L(p))$ is a chain for each $p \in D$.*

Proof. Let x and y be elements of $\phi(L(p))$. Then $x = \phi(q_1)$ and $y = \phi(q_2)$ where $\{q_1, q_1\} \subset L(p)$. Since $L(p)$ is a chain in D, it follows that $q_1 \leq_e q_2$ or $q_2 \leq_e q_1$, say $q_1 \leq_e q_2$. But this implies that $x \leq_D y$ since $q_1 \in \phi^{-1}(x)$ and $q_2 \in \phi^{-1}(y)$. Similarly, if $q_2 \leq_e q_1$, then $y \leq_D x$. It follows that $\phi(L(p))$ is a chain.

Lemma 3.10. *If $x \in X - A(X)$ then $L(x)$ is a chain.*

Proof. Suppose that $x \in X - A(X)$. Then Lemma 3.8 implies that

$$L(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(L(p)).$$

But $x \notin A(X)$ implies that $\phi^{-1}(x)$ is a single point. Thus $L(x) = \phi(L(\phi^{-1}(x))$ which is a chain by Lemma 3.9.

Lemma 3.10 shows that $L(x)$ is a chain for each point of X which is not a member of $A(X)$. Our next lemma relates the cardinality of the set $\phi^{-1}(x)$ to that of a maximal antichain of $L(x)$ for each $x \in A(X)$.

Lemma 3.11. *Let $x \in A(X)$. Then $\phi^{-1}(x)$ is finite if and only if every antichain of $L(x)$ is finite. Furthermore, if $|\phi^{-1}(x)| = n < \infty$, and A is an antichain of $L(x)$, then $|A| \leq n$.*

Proof. Suppose $|\phi^{-1}(x)| = n < \infty$ and let A be a subset of $L(x)$ such that $|A| > n$. It follows from Lemma 3.8 that $L(x) = \bigcup_{p\in\phi^{-1}(x)} \phi(L(p))$. Since $|\phi^{-1}(x)| = n$, it must be the case that there exists $p \in \phi^{-1}(x)$ such that $\phi(L(p))$ contains at least two members of A, say $\{a_1, a_2\} \subset A \cap \phi(L(p))$. Furthermore, Lemma 3.9 shows that $\phi(L(p))$ is a chain. Therefore a_1 and a_2 are comparable, which implies that A is not an antichain. It follows that every antichain of $L(x)$ is finite, and that if A is an antichain of $L(x)$, and if $|\phi^{-1}(x)| = n$, then $|A| \leq n$.

Now suppose that $\phi^{-1}(x)$ is infinite. Let p_0 be a limit point of $\phi^{-1}(x)$, and let $\{p_n\}$ be a sequence in $\phi^{-1}(x)$ which converges to p_0. Let $P = \{p_n\}$.

We may assume that $p_0 \neq p_n$ for all $n \in \mathbb{N}$, so that $p_0 \notin P$. Note that P is a subset of $\phi^{-1}(x)$ which is contained in $\operatorname{Max}(D)$. Now we will establish the following claim:

Claim. For each $p_k \in P$, there exists $z_k <_e p_k$ such that $M(z_k) \cap P = \{p_k\}$.

Suppose that $p_k \in P$. Let $s_k = \sup(L(p_0) \cap L(p_k))$. Then s_k is an element of $L(p_0) \cap L(p_k)$ by Theorem 2.1. Note that $s_k <_e p_k$, for if $s_k = p_k$, then $p_k \in L(p_0)$ which implies that $p_0 = p_k$, a contradiction. Therefore we may choose t_k such that $s_k <_e t_k <_e p_k$. Then $p_0 \notin M(t_k)$, for if $t_k \leq_e p_0$, then $t_k \in L(p_0) \cap L(p_k)$ which implies that $t_k \leq_e s_k$ and we chose t_k so that $s_k <_e t_k$. It follows that $p_0 \in D - M(t_k)$ which is open. Therefore there exist $N \in \mathbb{N}$ such that if $n \geq N$ then $p_n \in D - M(t_k)$.

Now, for each $j < N$, $j \neq k$, let $s_j = \sup(L(p_j) \cap L(p_k))$. Then s_j is an element of $L(p_j) \cap L(p_k)$ by Theorem 2.1. Note that $s_j <_e p_k$ since $\{p_k, p_j\}$ is a subset of $\phi^{-1}(x) \subset E(D)$ so that p_j and p_k are not comparable. Let $S = \{s_1, \cdots, s_{N-1}, t_k\}$. Since $S \subset L(p_k)$, it follows that S is a chain. Let s be the largest element in S. It follows that

$$t_k \leq_e s \tag{1}$$

and

$$s_j \leq_e s \quad \text{for each } s_j \in S. \tag{2}$$

Also, since $s_j <_e p_k$ for each $s_j \in S$ and $t_k <_e p_k$, it follows that $s <_e p_k$. Therefore there exists z_k such that

$$s <_e z_k <_e p_k. \tag{3}$$

We will show that z_k is the desired element of D.

First note that $z_k <_e p_k$ by choice. Now suppose there exists $p_n \in M(z_k) \cap P$ such that $p_n \neq p_k$. If $n \geq N$, then $p_n \in D - M(t_k)$. But p_n is an element of $M(z_k)$ implies that $z_k \leq_e p_n$. Furthermore, (1) and (3) show that $t_k \leq_e s <_e z_k$ which implies that $t_k \leq_e s <_e z_k \leq_e p_n$. But this implies that $p_n \in M(t_k)$, a contradiction. If $n < N$, then $z_k \leq_e p_n$ and $z_k <_e p_k$ by (3), so that $z_k \in L(p_k) \cap L(p_n)$. Also $s_n = \sup(L(p_k) \cap L(p_n))$ which implies that $z_k \leq_e s_n$. But (2) and (3) show that $s_n \leq_e s <_e z_k$ so that $s_n <_e z_k$. We have again reached a contradiction. Therefore it must be the case that $M(z_k) \cap P = \{p_k\}$ and the claim is established.

We will now construct an infinite antichain A of D such that $\phi(A)$ is an infinite antichain of $L(x)$. For each $p_n \in P$, there exists a point z_n such that $z_n <_e p_n$ and $M(z_n) \cap P = \{p_n\}$. Let $A = \{z_n : n \in \mathbb{N}\}$. Suppose that $z_j <_e z_k$ for some z_j and z_k in A, $j \neq k$. Then $z_j <_e p_k$ so that $p_k \in M(z_j)$. But this contradicts the fact that $M(z_j) \cap P = \{p_j\}$. Therefore A is an antichain of D.

Finally, consider $\phi(A)$. First note that if $\phi(z_n) \in \phi(A)$ then $\phi(z_n) \leq_D x$ since $z_n \leq_e p_n$, $z_n = \phi^{-1}(\phi(z_n))$, and $p_n \in \phi^{-1}(x)$. Therefore $\phi(A) \subset L(x)$. Also note that since $A \subset D - E(D)$ it follows that ϕ is a bijection between A and $\phi(A)$ and so $\phi(A)$ is infinite. To see that $\phi(A)$ is an antichain, suppose there exist $\phi(z_j)$ and $\phi(z_k)$ in $\phi(A)$ such that $\phi(z_j) \leq_D \phi(z_k)$. Since $\phi^{-1}(\phi(z_j)) = \{z_j\}$ and $\phi^{-1}(\phi(z_k)) = \{z_k\}$, it must be the case that $\phi(z_j) =$

$\phi(z_k)$. Thus $\phi(A)$ is an infinite antichain of $L(x)$ and this completes the proof of Lemma 3.11.

The following theorem summarizes the properties of the partial order just constructed for Peano continua.

Theorem 3.12. *Let X be a Peano continuum and $\phi : D \to X$ where D is a dendrite and ϕ is a map such that $\phi|_{D-E(D)}$ is one-to-one and $\phi(D - E(D)) \cap \phi(E(D) = \varnothing$. Let $A(X) = \{x : |\phi^{-1}(x)| > 1\}$. Define a binary relation $\leq_D$ on X by $x \leq_D y$ if and only if there exist $p \in \phi^{-1}(x)$ and $q \in \phi^{-1}(y)$ such that $p \leq_e q$ where $\leq_e$ is the cutpoint partial order on D with respect to some basepoint e. Then $\leq_D$ is a continuous, order dense partial order satisfying the following conditions:*

(i) $A(X) \subset \mathrm{Max}(X)$,
(ii) *the point $\phi(e)$ is a zero and if $x \in X - A(X)$, then $L(x)$ is a chain,*
(iii) *if $x \in A(X)$ and A is a maximal antichain of $L(x)$, then $|A| = n < \infty$ if and only if $|\phi^{-1}(x)| = n < \infty$.*

The next lemma is needed for the order-theoretic characterizations of Peano continua.

Lemma 3.13. *Let X be a compact Hausdorff space which admits a continuous, order dense partial order $\leq$ with zero e, and suppose that $A(X) \subset \mathrm{Max}(X)$ such that $L(x)$ is a chain for each $x \in X - A(X)$. Then $L(x)$ is connected for each $x \in X - A(X)$, and X and $X - A(X)$ are both connected.*

Proof. Since X contains a zero it follows that $\mathrm{Min}(X)$ is a single point, and so Lemma 3 of [W2] shows that X is connected.

Now suppose $x \in X - A(X)$. Then $L(x)$ is a compact, order dense POTS, which is actually a chain. Furthermore, $\mathrm{Min}(L(x)) = \{e\}$. It follows from Lemma 3 of [W2] that $L(x)$ is connected. This implies that $X - A(X)$ is connected since

$$X - A(X) = \bigcup_{x \in X - A(X)} L(x),$$

each $L(x)$ is connected, and they all have the point e in common.

We are now ready for the theorem which characterizes Peano continua order theoretically.

Theorem 3.14. *Let X be a compact metric space. Then X is a Peano continuum if and only if X admits partial order $\leq$, and $A(X) \subset \mathrm{Max}(X)$ satisfying the following conditions:*

(i) *$\leq$ is order dense,*
(ii) *$\leq$ is continuous,*
(iii) *X contains a zero, and if $x \in X - A(X)$, then $L(x)$ is a chain,*
(iv) *$X - A(X)$ is rim-compact,*
(v) *$M(x) - (\{x\} \cup A(X))$ is an open subset of $X - A(X)$ for each x in $X - A(X)$.*

Proof. Suppose X is a Peano continuum. Then $X = \phi(D)$ where D is a dendrite and ϕ is a map such that $\phi|_{D-E(D)}$ is one-to-one. Define $\leq_D$ as in Theorem 3.12, and let $A(X) = \{x : |\phi^{-1}(x)| > 1\}$. Then Theorem 3.12 shows that $\leq_D$ is a continuous, order dense partial order satisfying condition (iii) and that $A(X) \subset \mathrm{Max}(X)$. Furthermore, Lemma 3.4 shows that $X - A(X)$ is regular and therefore rim-compact. Also, if $x \in X - A(X)$, then $|\phi^{-1}(x)| = 1$ and $M(x) = \phi(M(\phi^{-1}(x)))$ by Lemma 3.8 and so

$$(*) \qquad M(x) - (\{x\} \cup A(X)) = \phi(M(\phi^{-1}(x)) - (\{\phi^{-1}(x)\} \cup \phi^{-1}(A(X)))).$$

Since $M(\phi^{-1}(x)) - \{\phi^{-1}(x)\}$ is an open subset of D, it follows that $M(\phi^{-1}(x)) - (\{\phi^{-1}(x)\} \cup \phi^{-1}(A(X)))$ is an open subset of $D - \phi^{-1}(A(X))$. It is easy to check that $\phi|_{D-\phi^{-1}(A(X))}$ is a homeomorphism of $D - \phi^{-1}(A(X))$ onto $X - A(X)$. Therefore, $(*)$ shows that $M(x) - (\{x\} \cup A(X))$ is an open subset of $X - A(X)$ so that condition (v) of the theorem is satisfied.

For the converse, suppose that X admits a partial order $\leq$ and $A(X) \subset \mathrm{Max}(X)$ satisfying conditions (i)–(v). Then condition (iii) and Lemma 3.13 guarantee that X and $X - A(X)$ are connected.

It is a straightforward check that the partial order $\leq$ when restricted to the space $X - A(X)$, satisfies the following conditions:

(a) $\leq$ is order dense,
(b) if $x \in X - A(X)$, then $M(x)$ is closed in $X - A(X)$,
(c) if $x \in X - A(X)$ then $M(x) - \{x\}$ is open in $X - A(X)$,
(d) if $x \neq y$, then $L(x) - \{x\} \neq L(y) - \{y\}$.

By Theorem 11 in [W6], these conditions imply that $X - A(X)$ is dendritic. Furthermore, condition (iv) states that $X - A(X)$ is rim-compact.

It is shown in [P] that every rim-compact, dendritic space has a unique dendritic compactification. Let D be the unique dendritic compactification of $X - A(X)$. Note that D is a dendrite since it is a compact dendritic space.

Since $X - A(X)$ is dense in D, and X is a compact metric space, we may extend the identity map, $id : (X - A(X)) \subset D \to X$, to a continuous map $\phi : D \to X$. It follows that X is the image under a closed map of the locally connected space D, and hence X is locally connected. This completes the proof of Theorem 3.14.

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