MAXIMAL IDEALS IN LAURENT POLYNOMIAL RINGS

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(Communicated by Louis J. Ratliff, Jr.)

Abstract. We prove, among other results, that the one-dimensional local domain \( A \) is Henselian if and only if for every maximal ideal \( M \) in the Laurent polynomial ring \( A[T, T^{-1}] \), either \( M \cap A[T] \) or \( M \cap A[T^{-1}] \) is a maximal ideal. The discrete valuation ring \( A \) is Henselian if and only if every pseudo-Weierstrass polynomial in \( A[T] \) is Weierstrass. We apply our results to the complete intersection problem for maximal ideals in regular Laurent polynomial rings.

1. Introduction

Let \( A \) be a commutative Noetherian ring with identity. Let \( R \) denote the Laurent polynomial ring \( A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}] \), where \( X_i \) and \( Y_i \) are distinct indeterminates over \( A \). Let \( M \) be a maximal ideal in \( R \). Let

\[
M_1 = M \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m]
\]

and

\[
M_2 = M \cap A[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}].
\]

The content of this paper is the investigation of the following question.

Question. When is \( M_1 \) or \( M_2 \) a maximal ideal? In other words, when do maximal ideals in \( R \) come from maximal ideals in \( A[X_1, \ldots, X_n, Y_1, \ldots, Y_m] \) or \( A[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}] \)?

We provide a complete answer to this question. With the above setup of notations, we prove that for every maximal ideal \( M \) in \( R \), \( M_1 \) or \( M_2 \) is a maximal ideal if and only if \( A/P \) is a Henselian ring for every \( G \)-ideal \( P \) in \( A \). As a consequence, we prove that the one-dimensional local domain \( A \) is Henselian if and only if for every maximal ideal \( M \) in the Laurent polynomial ring \( A[T, T^{-1}] \), either \( M \cap A[T] \) or \( M \cap A[T^{-1}] \) is a maximal ideal, and thus we answer a question suggested in [12, Remark, p. 689].

Since a quotient of a Henselian ring is Henselian, it follows that if \( A \) is Henselian then for every maximal ideal \( M \) in \( R \), either \( M_1 \) or \( M_2 \) is maximal. Abhyankar, Heinzer, and Wiegand [1] have produced an example of a non-Henselian ring \( A \) such that \( A/P \) is Henselian for every \( G \)-ideal \( P \) in \( A \).

Received by the editors October 5, 1990 and, in revised form, January 10, 1991.

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For terminology my standard source is Nagata [10]. All the rings we consider are commutative Noetherian with identity. The dimension of a ring means the Krull dimension, and all rings are assumed to have finite dimension.

2. Preliminaries

Let us recall that in the ring $A$ a prime ideal $P$ is called a $G$-ideal if $P$ is the contraction of a maximal ideal in the polynomial ring $A[T]$ (see [8]). It is well known (and is easy to prove) that a prime ideal $P$ in $A$ is a $G$-ideal if and only if $A/P$ is a semilocal domain of dimension $\leq 1$. The ring $A$ is, by definition, a Hilbert ring if every $G$-ideal in $A$ is maximal. Finitely generated algebras over Hilbert rings are Hilbert rings. Thus, in the case when $A$ is a Hilbert ring, we observe that both $M_1$ and $M_2$ are maximal ideals.

Let $P = M \cap A$. A generalized version of a theorem of Artin and Tate [2, Theorem 4] amounts to the following: If $B$ is a Noetherian domain such that some finitely generated $B$-algebra is a semilocal domain of dimension $\leq 1$, then $B$ is semilocal of dimension $\leq 1$. For a proof of this statement, see [6, 15.1]. In the situation under consideration, we have that the field $R/M$ is a finitely generated $(A/P)$-algebra, so we conclude that $A/P$ is a semilocal domain of dimension $\leq 1$. Hence $P$ is a $G$-ideal.

The following couple of theorems are crucial to the proofs of our main results.

Theorem A [12, Theorem 2.2]. Let $A$ be a local domain of dimension one. Then $A$ is Henselian if and only if $A'$ (the derived normal ring of $A$) is a discrete valuation ring such that if $f \in A'[T]$ is an irreducible polynomial of degree $\geq 1$, then either $f$ is monic or $f(0)$ is a unit in $A'$.

Lacking a proper reference, I choose to give a proof of the following needed result. Recall that a ring $A$ is said to satisfy the first chain condition for prime ideals if every maximal chain of prime ideals in $A$ has length equal to the dimension of $A$ [10, p. 123].

Theorem B. Let $A$ be a Noetherian domain of dimension $d$. Assume that $A$ satisfies the first chain condition for prime ideals. Let $R$ be the Laurent polynomial ring $A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$. Then the height of every maximal ideal in $R$ is $d + n + m$ or $d + n + m - 1$.

Proof. Let $M$ be a maximal ideal in $R$ and $P = M \cap A$. If $P$ is a maximal ideal in $A$, then $M/PR$ is a maximal ideal in the affine domain $(A/P)[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$ over the field $A/P$. Hence $ht(M/PR) = n + m$. By assumption, $ht(P) = d$. Thus it follows that $M$ has height $d + n + m$. If $P$ is not maximal then $dim(A/P) = 1$. By the assumption that $A$ satisfies the first chain condition for prime ideals, we have $ht(P) = d - 1$. $M/PR$ is a maximal ideal in the domain $(A/P)[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$ such that $(M/PR) \cap (A/P) = (0)$. Going through the quotient field of $A/P$, we observe that $ht(M/PR) = n + m$. Then, $ht(M) \geq ht(P) + ht(M/PR) = d - 1 + n + m$. On the other hand, let us observe that for any prime ideal $Q$ in $R$, $ht(Q) \leq ht(Q \cap A) + n + m$. Hence $ht(M) = d - 1 + n + m$. 

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3. The main results

We first prove a simple lemma.

Lemma 1. Let $A \rightarrow B$ be an integral extension of domains. Then for every maximal ideal $M$ in the ring $A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$, either $M_1 = M \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$ or $M_2 = M \cap A[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}]$ is a maximal ideal if and only if for every maximal ideal $N$ in the ring $B[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$, either $N_1 = N \cap B[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$ or $N_2 = N \cap B[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}]$ is a maximal ideal.

Proof. We only prove one part leaving the other for the reader. Let us assume that either $M_1$ or $M_2$ is maximal for every maximal ideal $M$. Let $N$ be a maximal ideal in $B[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$. Let $M = N \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$. Without loss of generality, we assume that $M_1 = M \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$ is a maximal ideal. Let $N_1 = N \cap B[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$. We show that $N_1$ is a maximal ideal. Observe $M_1 = N_1 \cap B[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$. Since $B[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$ is integral over $A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$, and the prime ideal $N_1$ contracts to the maximal ideal $M_1$ of $A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$, we have that $N_1$ is maximal.

We now prove

Theorem 1. Let $A$ be a ring such that $A/P$ is a Henselian ring for every $G$-ideal $P$ in $A$. Let $M$ be a maximal ideal in the Laurent polynomial ring $R = A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$. Then either $M_1 = M \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Y_m^{-1}]$ or $M_2 = M \cap A[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}]$ is a maximal ideal.

Proof. Let $P = M \cap A$. The first reduction is that we go modulo $P$, and assume that $M \cap A = (0)$. Then $A$ is a semilocal domain of dimension $\leq 1$. If $\dim(A) = 0$ then $A$ is a field, and we have that both $M_1$ and $M_2$ are maximal ideals. So, we assume that $\dim(A) = 1$. Using the given hypothesis, we have that $A$ is a Henselian local domain.

We now proceed by induction on $m$. Let $A'$ denote the derived normal ring of $A$. Since $A$ is a Henselian local domain of dimension one, $A'$ is a Henselian discrete valuation ring. By Lemma 1, we may pass onto $A'$ to prove the theorem. Thus, we assume that $A$ is a Henselian discrete valuation ring with $\pi \in A$ as a uniformizing parameter. By virtue of Theorem B, we have that $\text{ht}(M) = 1 + n + m$ or $n + m$. Since $M \cap A = (0)$, it is the case that $\text{ht}(M) = n + m$. We note that $\text{ht}(M_1) = \text{ht}(M_2) = n + m$. Set $Y = Y_1$. Let $Q = M \cap A[Y] = M_1 \cap A[Y]$. Then $Q$ is a prime ideal of height one in $A[Y]$, and $M_1/Q$ is a prime ideal of height $n + m - 1$ in the polynomial ring $(A[Y]/Q)[X_1, \ldots, X_n, Y_2, \ldots, Y_m]$.

If $Q$ is a maximal ideal then we observe that $M_1/Q$ is a maximal ideal. Hence $M_1$ is a maximal ideal. Suppose that $Q$ is not a maximal ideal. Since $A[Y]$ is a unique factorization domain, we have that $Q = (f)$, where $f$ is an irreducible polynomial in $A[Y]$. By Theorem A, either $f$ is monic or $f(0)$ is a unit in $A$. Changing $A[Y]$ to $A[Y^{-1}]$, if necessary, there is no loss of generality in assuming that $f$ is a monic polynomial in $A[Y]$.

Case (i). $f(0) \in \pi A$. Let $f = Y^t + a_1 Y^{t-1} + \cdots + a_t$. Since $f$ is an
irreducible monic polynomial in $A[Y]$ with $a_i = f(0) \in \pi A$ and $(A, \pi)$ is a Henselian discrete valuation ring, we must have that each $a_i \in \pi A$; otherwise $f$ modulo $\pi A$ would factor as a product of two comaximal monics yielding a nontrivial factorization of $f$ in $A[Y]$. Let $Q' = M \cap A[Y^{-1}]$. Then $Q'$ is generated by $g = 1 + a_1 T + \cdots + a_l T^l$, where $T = Y^{-1}$. At this point, we make the observation that $g$ is not contained in any maximal ideal of height two in $A[Y^{-1}]$; this is because any maximal ideal of height two in $A[Y^{-1}]$ contains $\pi$, and thus it is co-maximal to $g$. Thus we have that $Q'$ is a maximal ideal in $A[Y^{-1}]$. By an argument similar to the one given earlier, we conclude that $M_2$ is a maximal ideal.

Case (ii). $f(0) \notin \pi A$. Then $(f, Y) = A[Y]$, and $f$ is monic in $Y$. Hence we have that the integral domain $A_1 = A[Y]/fA[Y] = A[Y, Y^{-1}]/fA[Y, Y^{-1}]$ is an integral extension of $A$. Since $A$ is a Henselian domain of dimension one, we have that $A_1$ is a Henselian domain of dimension one [10, 43.13]. We go modulo $fA[Y, Y^{-1}]$ and obtain that $M' = M/fA[Y, Y^{-1}]$ is a maximal ideal in $A_1[X_1, \ldots, X_n, Y_2, Y_2^{-1}, \ldots, Y_m, Y_m^{-1}]$. By induction on $m$, either $M_3 = M' \cap A_1[X_1, \ldots, X_n, Y_2, Y_2^{-1}, \ldots, Y_m, Y_m^{-1}]$ or $M_4 = M' \cap A_1[X_1, \ldots, X_n, Y_2^{-1}, \ldots, Y_m^{-1}]$ is a maximal ideal.

Let us note that $M_1/fA[Y] = M_1/Q = M_3$ and $M_4 = M_2/Q'$. Consequently, $M_1$ or $M_2$ is a maximal ideal. The proof is complete.

To establish the converse of Theorem 1, we need the following lemma.

**Lemma 2.** Let $A$ be a one-dimensional semilocal domain. Assume that for every maximal ideal $M$ in the Laurent polynomial ring $A[T, T^{-1}]$, either $M \cap A[T]$ or $M \cap A[T^{-1}]$ is a maximal ideal. Then the derived normal ring of $A$ is local; in particular, $A$ is local.

**Proof.** Let us assume that $A'$, the derived normal ring of $A$, is not local. Since $A$ is a one-dimensional Noetherian semilocal domain, $A'$ is a semilocal Dedekind domain [10, 33.2, 33.10]. Hence $A'$ is a principal ideal domain with only a finite number of prime ideals [14, p. 12]. Let $p_1, p_2, \ldots, p_r$ be all the distinct (nonassociate) primes in $A'$. By assumption we have $r > 1$. Let $a_0 = p_1 p_2 \cdots p_r$. For each $i = 1, 2, \ldots, r$, let $a_i = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_r$. In the polynomial ring $A'[T]$, set $f = a_0 + a_1 T + \cdots + a_r T^r$. By Eisenstein’s criterion, $f$ is an irreducible polynomial in $A'[T]$. Since $fA'[T]$ is not a maximal ideal, any maximal ideal in $A'[T]$ containing $fA'[T]$ will have height two. Then such a maximal ideal must contain some $p_i$ and hence $T$. So the maximal ideals in $A'[T]$ that contain $fA'[T]$ are precisely $(p_i, T)$, $i = 1, 2, \ldots, r$. Hence $M = fA'[T, T^{-1}]$ is a maximal ideal in $A'[T, T^{-1}]$, and we have that neither $M \cap A'[T]$ nor $M \cap A'[T^{-1}]$ is a maximal ideal. By letting $P = M \cap A[T, T^{-1}]$, we observe that $P$ is a maximal ideal in $A[T, T^{-1}]$ such that its contractions to $A[T]$ and $A[T^{-1}]$ are not maximal, a contradiction. Hence $A'$ is local.

We now prove

**Theorem 2.** Let $A$ be a ring such that for every maximal ideal $M$ in the Laurent polynomial ring $A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$, either $M_1 = M \cap A[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ or $M_2 = M \cap A[X_1, \ldots, X_n, Y_1^{-1}, \ldots, Y_m^{-1}]$ is a maximal ideal. Then $A/P$ is a Henselian ring for every $G$-ideal $P$ in $A$.

**Proof.** Let $P$ be a $G$-ideal in $A$. If $P$ is maximal then $A/P$ is trivially...
Henselian. So we assume that \( P \) is not maximal. Then \( A/P \) is a one-dimensional semilocal domain. Note that the hypothesis in the statement of the theorem remains valid when we replace \( A \) by \( A/P \) (pay attention to only the maximal ideals containing \( P \)). So we replace \( A \) by \( A/P \) and then prove that \( A \) is Henselian. By Lemma 2, \( A \) is local. To prove that \( A \) is Henselian, it suffices to prove that any domain \( B \) that is an integral extension of \( A \) is quasilocal \([10, 43.12]\). This is equivalent to proving that any domain \( B \) that is a finite \( A \)-module is local. Let \( B \) be a domain that is a finite \( A \)-module. By Lemma 1, \( B \) enjoys the hypothesis assumed for \( A \). By Lemma 2, \( B \) is local. Hence \( A \) is Henselian.

Since the zero ideal is a \( G \)-ideal in a local domain of dimension one, we have

**Corollary 1.** Let \( A \) be a one-dimensional local domain. Then \( A \) is Henselian if and only if every maximal ideal in the Laurent polynomial ring \( A[T, T^{-1}] \) contracts to a maximal ideal in \( A[T] \) or to a maximal ideal in \( A[T^{-1}] \).

Heinzer, Lantz, and Wiegand have independently proved Corollary 1.

### 4. Applications

Let \( (A, \mathfrak{m}) \) be a quasi-local ring. In [12] we defined a polynomial \( f \) in \( A[T] \) to be pseudo-Weierstrass if \((\mathfrak{m}, T)\) is the only maximal ideal in \( A[T] \) that contains \( f \).

Let \( (A, \mathfrak{m}) \) be a quasi-local ring. A monic polynomial \( f \in A[T] \) is called a Weierstrass polynomial if \( f = T^n + a_1T^{n-1} + \cdots + a_n \), where each \( a_i \in \mathfrak{m} \).

Clearly a Weierstrass polynomial is pseudo-Weierstrass. If \( A \) is a local domain of dimension at least two, then pseudo-Weierstrass polynomials in \( A[T] \) are precisely the Weierstrass ones, as was proved in [12, Proposition 3.1]. Let \( (A, \pi) \) be a discrete valuation ring such that the polynomial \( f = \pi T^2 + T + \pi \) is irreducible in the polynomial ring \( A[T] \). Then \((\pi, T)\) is the only maximal ideal in \( A[T] \) that contains \( f \). Hence \( f \) is a pseudo-Weierstrass polynomial that is not Weierstrass. So, in the case of a discrete valuation ring, when is every pseudo-Weierstrass polynomial Weierstrass? In [12, Theorem 3.6] it was proved that if \( A \) is a discrete valuation ring, then every pseudo-Weierstrass polynomial in \( A[T] \) is Weierstrass if and only if every maximal ideal in the Laurent polynomial ring \( A[T, T^{-1}] \) contracts to a maximal ideal in \( A[T] \) or in \( A[T^{-1}] \). Thus a combination of Corollary 1 and [12, Theorem 3.6] gives us the following.

**A1.** Let \( A \) be a discrete valuation ring. Then every pseudo-Weierstrass polynomial in \( A[T] \) is Weierstrass if and only if \( A \) is Henselian.

Let \( I \) be an ideal in a ring \( R \). We say that \( I \) is a complete intersection ideal if it can be generated as an \( R \)-module by \( \text{ht}(I) \) elements. A Noetherian ring \( R \) is called strongly regular if every maximal ideal of \( R \) is a complete intersection ideal \([6, p. 148]\). In [5, Theorem 2] it was shown that a polynomial extension of a regular Hilbert domain is strongly regular. Thus we have

**A2. A Laurent polynomial extension of regular Hilbert domain is strongly regular.**

Regular Hilbert domains exist in abundance. The rings of polynomial functions on nonsingular algebraic varieties are classical examples of such domains.
A really interesting way to get examples of regular Hilbert domains is the following. Start with a Noetherian ring $A$. Let $A[T]$ be the polynomial ring in one indeterminate over $A$, and let $A(T)$ denote the localization of $A[T]$ at the multiplicative set of all monic polynomials. Then $A(T)$ is a Hilbert ring. This result was the content of [4]. A very easy and nice way (following a suggestion of J. T. Stafford) to see this is as follows. Set $Y = 1/T$. Let $S$ be the multiplicative set $1 + YA[Y]$ in $A[Y]$. Throw $Y$ in the Jacobson radical by forming the ring $B = S^{-1}A[Y]$. It is easy to verify that $A(T) = B[1/Y]$; for details, see [9, p. 99]. Now the fact that $A(T)$ is a Hilbert ring is a consequence of the following beautiful application [7, 10.5.8] of the Principal Ideal Theorem of Krull: Let $B$ be a Noetherian ring, and let $a$ be a nonnilpotent element in the Jacobson radical of $B$. Then $B[1/a]$ is a Hilbert ring. Thus if we start with a regular ring $A$, then we have that $A(T)$ is a regular Hilbert domain. Let us now record the following.

A3. If $A$ is a regular ring then $A(T)[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$, with $n + m \geq 1$, is a strongly regular ring.

For A3, if $A$ is a regular locality (localization at a regular prime ideal of an affine algebra over a field) with infinite residue field or if $A$ is a formal power series ring over a field, then $n + m$ may be zero; this was proved in [13]. It is not known whether $A(T)$ is a strongly regular ring for any regular local ring $A$.

Let $A$ be a Henselian local ring such that polynomial extensions of $A$ are strongly regular (consequently, $A$ is a regular local ring). Then using Theorem 1, we have that Laurent polynomial extensions of $A$ are also strongly regular. For instance, it is known that if $A$ is a formal power series ring with coefficients in a field, then any polynomial extension of $A$ is a strongly regular ring [3, Theorem 3.1; 11, Theorem 2.2]. Thus, we have

A4 (cf. 15, Theorem 2.8). Let $A = k[[T_1, \ldots, T_d]]$, where $k$ is a field. Then the Laurent polynomial ring $A[X_1, \ldots, X_n, Y_1, Y_1^{-1}, \ldots, Y_m, Y_m^{-1}]$ is strongly regular.

Acknowledgment

I express my sincere thanks to Professor Sylvia Wiegand. Her interest in this work inspired me to obtain a complete answer to the question in the introduction.

References


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