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A SHORT PROOF OF THE Y. KATZNELSON'S AND L. TZAFRIRI'S THEOREM

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ABSTRACT. A short proof is given to the following theorem of Y. Katznelson and L. Tzafriri: Let T be a power-bounded operator in a Banach space E. Then $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$ if and only if $\sigma(T) \cap \{z \in \mathbb{C}: |z| = 1\} \subset \{1\}$.

Y. Katznelson and L. Tzafriri [2] proved the following theorem:

Theorem. *Let T be a power bounded operator on a Banach space E. Then $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$ if and only if $\sigma(T) \cap \{z \in \mathbb{C}: |z| = 1\} \subset \{1\}$.*

The 'only if' part of this theorem is simple. The proof of Y. Katznelson and L. Tzafriri for the 'if' part relies on some theorem of Tauberian type. In this note we present a short proof.

Proof. First we remark that it is enough to prove the following formerly weaker statement: if $\sigma(T) \cap \{z \in \mathbb{C}: |z| = 1\} \subset \{1\}$ then $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$, $\forall x \in E$ $(*)$.

Indeed, suppose that $(*)$ is proved. Let T be a power bounded operator on E such that the spectrum of T on the unit circle is at most a single point 1. Consider the Banach space $\mathscr{B}(E)$ of all bounded operators on E and the operator $\mathscr{T}$ on $\mathscr{B}(E)$ defined by: $\mathscr{T}S = TS$. Then it is clear that $\mathscr{T}$ is power bounded and has the same spectral property as T. Therefore, $\lim_{n\to\infty} \|\mathscr{T}^{n+1}S - \mathscr{T}^n S\| = 0 \forall S \in \mathscr{B}(E)$. Taking $S = I$, we get the desired conclusion.

Now let T be a power bounded operator such that the spectrum of T on the unit circle consists of at most one point 1. We can assume, without loss of generality, that $\|T\| \le 1$. Then the limit $l(x) = \lim_{n\to\infty} \|T^n x\|$ exists and is a seminorm in E. Put $L = \ker l$. If $L = E$, then the conclusion is obvious, and we can assume that $L \ne E$. It is clear that L is a closed invariant subspace of T, so that T induces in a natural way an operator $\widehat{T}$ on the quotient space $\widehat{E} = E/L$. The seminorm l induces a norm $\hat{l}$ on $\widehat{E}$, and since T is isometric in the seminorm l, the operator $\widehat{T}$ in the normed space $(\widehat{E}, \hat{l})$ is an isometry. From the obvious inequality $l(R_\lambda x) \le \|R_\lambda\| l(x)$,

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where $R_\lambda = (T-\lambda)^{-1}$, $\lambda \notin \sigma(T)$, it follows that $\sigma(\widehat{T}) \subset \sigma(T)$. Therefore, $\widehat{T}$ is an isometry with single peripheral spectrum at 1. From this it is easily seen that $\widehat{T}$ is an invertible isometry with single spectrum at 1. It is a well-known theorem of I. Gelfand that such an isometry must be the identity (see e.g. [1]). This means that $\lim_{n\to\infty} \|T^n(T-I)x\| = 0$, $\forall x \in E$. The proof is completed.

Remark. The referee has pointed out that a similar proof of the Katznelson-Tzafriri Theorem has been obtained by G. R. Allan and T. J. Ransford (see G. R. Allan and T. J. Ransford, *Power-dominated elements in a Banach algebra*, Studia Math. **94** (1989), 63–79). However, a closer look shows that two proofs are essentially different. Our method is shorter and more natural in the context of the operator theory. Moreover, it can be used to obtain important theorems on strong stability, while the method of G. R. Allan and T. J. Ransford does not lead to such results (see Yu. I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. **88** (1988), 37–42, and also Vũ Quôc Phóng, *Theorems of Katznelson-Tzafriri type for semigroups of operators*, J. Funct. Anal., in press). Also note that another proof of the Katznelson-Tzafriri Theorem, which uses a Tauberian theorem and is completely different from our proof, is given in G. R. Allan, A. G. O'Farrell, and J. Ransford, *A Tauberian theorem arising in operator theory*, Bull. London Math. Soc. **19** (1987), 537–545.

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