A NOTE ON A THEOREM OF J. DIESTEL AND B. FAIRES

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Dedicated to Professor M. Valdivia on the occasion of his sixtieth birthday

Abstract. Applying a property concerning certain coverings of $l_0^\infty(X, \mathcal{A})$ that always contain some elements that are barrelled and dense in $l_0^\infty(X, \mathcal{A})$, we generalize a localization theorem of M. Valdivia, relative to vector bounded finitely additive measures (Theorem 1), and obtain two different generalizations of a theorem of J. Diestel and B. Faires ensuring that certain finitely additive measures are countably additive (Theorems 2 and 3).

The original proof of the quoted theorem of Diestel and Faires uses a theorem of Rosenthal that is not required in our proof of Theorem 3. This avoids imposing over the Valdivia's $\Lambda_r$-spaces defining the measure range space, the condition that they do not contain a copy of $l^\infty$.

Introduction

From now onwards the word "space" will mean "locally convex Hausdorff space over the field $K$ of the real or complex numbers." We set $\mathcal{A}$ to denote a $\sigma$-algebra of subsets of a set $X$ and represent by $e(A)$ the characteristic function of the subset $A$ of $X$. Let $l_0^\infty(X, \mathcal{A})$ be the linear space generated by the family $\{e(A), A \in \mathcal{A}\}$ endowed with the topology defined by the supremum norm. As usual, we shall identify the space $B(\mathcal{A})$ of the bounded finitely additive scalar measures on $\mathcal{A}$ with the topological dual of the space $l_0^\infty(X, \mathcal{A})$, and the subspace of the countably additive scalar measures will be denoted by $M(\mathcal{A})$.

A space $E$ is dual locally complete [6] if $E'(\sigma(E', E))$ is locally complete. A space $E$ is $\Gamma_r$ [8] ($\Lambda_r$, [6]) if given any quasi-complete (locally complete) subspace $G$ of $E^*(\sigma(E^*, E))$ such that $G$ meets $E'$ in a dense subspace of $E'(\sigma(E', E))$, $G$ contains $E'$. $B_r$-complete spaces are $\Gamma_r$, and reflexive Banach spaces and Fréchet-Schwartz spaces provide some simple examples of $\Lambda_r$-spaces. For simplicity we introduce the following definition.

Definition. Given any positive integer $p$, a countable family of subspaces $W = \{L_{m_1m_2...m_r}, m_r \in \mathbb{N}, 1 \leq r \leq s \leq p\}$ of a linear space $L$ is a $p$-net in...
If the sequence \{L_{m_s}, m_s \in \mathbb{N}\} is increasing and covers \(L\) and for each \(s \in \{2, \ldots, p\}\), \(\{L_{m_{s-1} \ldots m_1} m_s \in \mathbb{N}\}\) is increasing and covers \(L_{m_{s-1} \ldots m_1} \).

We shall denote by \(W_p\) the family \(\{L_{m_{s-1} \ldots m_1} m_s \in \mathbb{N}, 1 \leq i \leq p\}\). In [3, Theorem 1] we have shown that if \(W\) is a \(p\)-net in \(l_0^\infty(X, \mathcal{A})\), then there exists some \(L_{m_{s-1} \ldots m_1} \) that is a dense and barrelled subspace of \(l_0^\infty(X, \mathcal{A})\). This result for \(p = 1\) has been obtained by M. Valdivia in [7, Theorem 1] showing that \(l_0^\infty(X, \mathcal{A})\) is suprabarrelled.

From the suprabarrelledness of \(l_0^\infty(X, \mathcal{A})\), the following two results have been derived.

(a) Let \(\mu\) be a bounded additive measure from a \(\sigma\)-algebra \(\mathcal{A}\) on \(X\) into a space \(E\). Let \(\{F_n, n = 1, 2, \ldots\}\) be an increasing sequence of \(\Gamma_r\)-spaces covering a space \(F\). If \(f: E \to F\) is a linear mapping with closed graph, there is a positive integer \(q\) such that \(f\mu\) is a \(F_q\)-valued bounded finite additive measure on \(\mathcal{A}\) [7, Theorem 4].

(b) Let \(\mu\) be a finitely additive measure on \(\mathcal{A}\) with values in \(E\), and let \(H\) be a \(\sigma(E', E)\) total subset of \(E'\) such that \(u\mu\) is a countably additive measure for each \(u \in H\). If \(E\) is a countable inductive limit of \(\Gamma_r\)-complete spaces that do not contain \(l_\infty\), then \(\mu\) is a countable additive measure [5, 9.4, p. 367].

This last result extends a well-known theorem of J. Diestel and B. Faires [1, Theorem 1.1].

Our previously quoted result of [3] enables us to generalize results (a) and (b) in Theorems 1 and 2 below. Besides, Theorem 2 has suggested to us a new generalization of the Diestel-Faires theorem, avoiding the condition that the range spaces do not contain a copy of \(l_\infty\).

\section*{Results}

\textbf{Theorem 1.} Let \(\mu\) be a bounded additive measure on \(\mathcal{A}\) with values in a space \(E\). Suppose that \(F\) is a space with a \(p\)-net \(W\) such that each \(L \in W_p\) has a locally convex topology \(\mathcal{T}_L\) stronger than that induced by \(F\), under which \(L(\mathcal{T}_L)\) is a \(\Gamma_r\)-space. If \(f\) is a linear mapping from \(E\) into \(F\) with closed graph, then there exists a \(G \in W_p\) such that \(f\mu\) is a \(G(\mathcal{T}_G)\)-valued bounded finitely additive measure.

\textit{Proof.} As \(\mu\) is bounded, the mapping \(S: l_0^\infty(X, \mathcal{A}) \to E\), such that \(S(e(A)) = \mu(A)\) for every \(A \in \mathcal{A}\), is continuous, and therefore the linear map \(T = fS\) has closed graph. By [3, Theorem 1], there is some \(G \in W_p\) such that \(H = T^{-1}(G)\) is dense in \(l_0^\infty(X, \mathcal{A})\) and barrelled.

According to Theorems 1 and 14 of [8], the restriction of \(T\) to \(H\) admits a continuous extension \(U\) in \(l_0^\infty(X, \mathcal{A})\) with values in \(G\). As \(T\) has closed graph, \(T = U\).

\textbf{Theorem 2.} Let \(\mu\) be a finitely additive measure on \(\mathcal{A}\) with values in a space \(E\), and let \(H\) be a \(\sigma(E', E)\)-total subset of \(E'\). Suppose that \(E\) has a \(p\)-net \(W\) such that in each \(L \in W_p\) there exists a locally convex topology \(\mathcal{T}_L\) finer than that induced by \(E\), under which \(L(\mathcal{T}_L)\) is a sequentially complete \(\Gamma_r\)-space not containing any copy of \(l_\infty\). If \(u\mu\) is a countably additive measure for each \(u \in H\), then there exists a \(G \in W_p\) such that \(\mu\) is a \(G(\mathcal{T}_G)\)-valued countably additive vector measure.
Proof. Let $F$ denote the linear hull of $H$. The mapping $S$ from $l_0^\infty(X, \mathcal{A})$ into $E$, such that $S(e(A)) = \mu(A)$ for every $A \in \mathcal{A}$, has closed graph, since by hypothesis $u\mu \in M(\mathcal{A})$ for every $u \in F$. By Theorem 1 of [3], there is some $G \in \mathcal{W}_p$ such that $K = S^{-1}(G)$ is dense in $l_0^\infty(X, \mathcal{A})$ and barrelled.

Again by Theorems 1 and 14 of [8], the restriction of $S$ to $K$ admits a continuous extension $U$ in $l^\infty(X, \mathcal{A})$ with values in $G(\mathcal{F}_G)$, so $\mu$ is strongly additive because of Rosenthal's theorem [2]. Now $\mu$ is countably additive, since if $\{A_n, n = 1, 2, \ldots\}$ is a sequence of pairwise disjoint subsets of $X$ belonging to $\mathcal{A}$, we have that $\mu(\bigcup\{A_n, n = 1, 2, \ldots\})$ is the only adherent point of the sequence $\{\sum(\mu(A_p), p = 1, 2, \ldots, n), n = 1, 2, \ldots\}$. In fact, $u\mu(\bigcup\{A_n, n = 1, 2, \ldots\}) = (\sum(u\mu(A_p), p = 1, 2, \ldots))$, for each $u \in H$.

In the last theorem, the subspaces $L$ belonging to $\mathcal{W}_p$ do not contain a copy of $l^\infty$ and are $\Gamma_r$ with a topology stronger than the induced one. Now we are going to prove that the former theorem can also be established if these subspaces $L$, provided with a topology stronger than the initial one, are of the class $\Lambda_r$ defined by Valdivia in [6].

We shall need the following well-known result of measure theory.

(c) Let $\{\lambda_n, n = 1, 2, \ldots\}$ be a sequence of elements of $M(\mathcal{A})$. If $\lim \lambda_n(A) = \lambda(A)$ for each $A \in \mathcal{A}$, then $\lambda \in M(\mathcal{A})$.

This result states that $M(\mathcal{A})(\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A})))$ is sequentially complete. The next proposition shows that $M(\mathcal{A})(\sigma(M(\mathcal{A}), E))$ is also sequentially complete when $E$ is a dense and barrelled subspace of $l_0^\infty(X, \mathcal{A})$. If $\mathcal{A}$ is infinite, there are in $M(\mathcal{A})(\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A})))$ bounded sequences without adherent point in $M(\mathcal{A})$. In fact, let $\{A_n, n \in \mathbb{N}\}$ be a sequence of nonempty pairwise disjoint elements of $\mathcal{A}$. Let $t_n$ be a point of $A_n$, and let $\delta_n$ be the Dirac measure on $t_n$. If $\lambda \in M(\mathcal{A})$, we can find a $p$ such that if $M = \bigcup\{A_n, n \geq p\}$, then $|\lambda(M)| < 1/2$, and therefore, for $n \geq p$ we have that $|\delta_n - \lambda| \geq \delta_n(M) - |\lambda(M)| > 1/2$. In [4], is shown that in $l^1(\sigma(l^1, l^\infty))$ there are bounded sequences without any adherent point.

**Proposition 1.** If $E$ is a dense and barrelled subspace of $l_0^\infty(X, \mathcal{A})$ then the space $M(\mathcal{A})(\sigma(M(\mathcal{A}), E))$ is sequentially complete and $l_0^\infty(X, \mathcal{A})$ is contained in the bounded closure of $E$ with respect to the dual pair $(E, M(\mathcal{A}))$.

**Proof.** The $E$-bounded subsets of $M(\mathcal{A})$ are $E$-equicontinuous, and since $E$ is dense in $l_0^\infty(X, \mathcal{A})$, they are also $l_0^\infty(X, \mathcal{A})$-equicontinuous. Hence the $E$-bounded subsets of $M(\mathcal{A})$ are $l_0^\infty(X, \mathcal{A})$-bounded. Thus, the second affirmation follows.

Now let $D$ be an absolutely convex subset of $M(\mathcal{A})$ that is bounded and closed under $\sigma(M(\mathcal{A}), E)$. Since $D$ is $\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A}))$-compact, we have that the topologies coincide in $D$ and also the translations invariant uniformities induced by $\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A}))$ and $\sigma(M(\mathcal{A}), E)$.

Hence any $\sigma(M(\mathcal{A}), E)$-Cauchy sequence $\{\lambda_n, n = 1, 2, \ldots\}$ in $M(\mathcal{A})$ is also $\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A}))$-Cauchy, and by result (c) there is some $\lambda \in M(\mathcal{A})$ such that $\lim \lambda_n = \lambda$ under $\sigma(M(\mathcal{A}), l_0^\infty(X, \mathcal{A}))$.

In particular, when $X = N$ and $\mathcal{A} = 2^N$, we have that if $E$ is any dense and barrelled subspace of $l_0^\infty$ then $l^1(\sigma(l^1, E))$ is weakly sequentially complete.

**Proposition 2.** Suppose that $W$ is a $p$-net contained in a space $F$, and let $f$ be a linear mapping from $l_0^\infty(X, \mathcal{A})$ into $F$ having closed graph in the product
$l^\infty(X, \mathcal{A})(\sigma(l^\infty(X, \mathcal{A}), M(\mathcal{A}))) \times F$. If each $L \in W_p$ has a locally convex topology $\mathcal{T}_L$ stronger than the induced by $F$ such that $L(\mathcal{T}_L)$ is a $\Lambda_r$-space, then there is some $G \in W_p$ containing the range of $f$ such that $f$ is weakly continuous with respect to the dual pairs $(l^\infty(X, \mathcal{A}), M(\mathcal{A}))$ and $(G, G(\mathcal{G}))$.

**Proof.** By Theorem 1 of [3] and Proposition 1, there is some $G$ such that $E: = f^{-1}(G)$ is dual locally complete with respect to the dual pair $(E, M(\mathcal{A}))$, and its bounded closure $\tilde{E}$ contains $l^\infty(X, \mathcal{A})$.

Let $g$ be the restriction of $f$ to $E$. As $g$ has closed graph in the product $E(\sigma(E, M(\mathcal{A}))) \times G(\mathcal{G})$, then by [6, Theorems 2 and 6] the mapping $g$ has a continuous extension $h$ from $l^\infty(X, \mathcal{A})(\sigma(l^\infty(X, \mathcal{A}), M(\mathcal{A})))$ with values in $G(\sigma(G, G(\mathcal{G})))$.

The mappings $f$ and $g$ are continuous, taking in $F$ a locally convex topology weaker than the initial one, and both coincide in $E$. This fact concludes the proof.

**Theorem 3.** Let $\mu$ be a countably additive measure on $\mathcal{A}$ with values in a space $E$. Suppose that $F$ is a space with a $p$-net $W$ such that each $L \in W_p$ has a locally convex topology $\mathcal{T}_L$ stronger than the induced by $F$ under which $L(\mathcal{T}_L)$ is a $\Lambda_r$-space. Suppose finally that $f$ is a linear mapping from $E$ into $F$ with closed graph. Then there exists a $G \in W_p$ such that $f\mu$ is a $G(\mathcal{G})$-valued countably additive measure.

**Proof.** The mapping $S: l^\infty(X, \mathcal{A})(\sigma(l^\infty(X, \mathcal{A}), M(\mathcal{A}))) \rightarrow E(\sigma(E, E'))$, such that $S(\mu(A)) = \mu(A)$ for every $A \in \mathcal{A}$ is continuous, since $u\mu \in M(\mathcal{A})$ for every $u \in E'$.

Then $T = fS$ has closed graph in $l^\infty(X, \mathcal{A})(\sigma(l^\infty(X, \mathcal{A}), M(\mathcal{A}))) \times F$.

By Proposition 2, there is some $G \in W_p$ such that $T(l^\infty(X, \mathcal{A})) \subset G$ and $T: l^\infty(X, \mathcal{A})(\sigma(l^\infty(X, \mathcal{A}), M(\mathcal{A}))) \rightarrow G(\sigma(G, G(\mathcal{G})))$ is continuous.

If $v \in G(\mathcal{G})'$, then the continuity of $T$ implies that $v f\mu \in M(\mathcal{A})$, and consequently the Orlicz-Pettis theorem implies that $f\mu$ is $\mathcal{G}$-countably additive.

**Corollary.** Let $\mu$ be an additive measure on $\mathcal{A}$ with values in a space $F$, and let $H$ be a $\sigma(F', F)$ total subset of $F'$. Suppose that $F$ has a $p$-net $W$ such that each $L \in W_p$ has a locally convex topology $\mathcal{T}_L$ stronger than that induced by $F$, under which $L(\mathcal{T}_L)$ is a $\Lambda_r$-space. We also suppose that $u\mu$ is countably additive for every $u \in H$. Then there exists a $G \in W_p$ such that $\mu$ is a $G(\mathcal{G})$-valued countably additive measure.

**Proof.** The corollary follows directly from Theorem 3 and Orlicz-Pettis theorem taking $E = F(\sigma(F, (H)))$ and $f$ the identity map on $E$.

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**References**


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