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A NOTE ON KRYLOV-TSO'S PARABOLIC INEQUALITY

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ABSTRACT. We show that if u is a solution to $\sum_{i,j=1}^{n} a_{ij}(x,t)D_{ij}u(x,t) - D_t u(x,t) = \phi(x)$ on a cylinder $\Omega_T = \Omega \times (0, T)$, where Ω is a bounded open set in $\mathbf{R}^n$, $T > 0$, and u vanishes continuously on the parabolic boundary of Ω_T. Then the maximum of u on the cylinder is bounded by a constant C depending on the ellipticity of the coefficient matrix $(a_{ij}(x,t))$, the diameter of Ω, and the dimension n times the L^n norm of ϕ in Ω.

INTRODUCTION

Let $L = \sum_{i,j=1}^{n} a_{ij}(x,t)D_{ij} - D_t$ be a parabolic operator with bounded measurable coefficients defined on a cylinder $\Omega_T = \Omega \times (0, T)$, where Ω is a bounded open set in $\mathbf{R}^n$, T is a positive real number, $a_{ij}(x,t) = a_{ji}(x,t)$ for all x in Ω, real t, and $i, j = 1, \ldots, n$, and where there exists $\lambda > 0$ so that the following holds,

$$(1)\qquad \lambda^{-1}|\xi|^2 \geq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega,\ t \in \mathbf{R},\ \text{and } \xi \in \mathbf{R}^n.$$

In [5, 6] it is shown that if $u \in C(\overline{\Omega}_T)$ is smooth in Ω_T with $u = 0$ on the parabolic boundary of Ω_T, i.e., $\partial_p\Omega_T = \partial\Omega \times [0, T) \cup \Omega \times \{0\}$ and $|Lu| \leq f(x,t)$ on Ω_T the following holds,

$$(2)\qquad \sup_{\Omega_T} |u| \leq C\delta(\Omega)^{n/(n+1)} \|f\|_{L^{n+1}(\Omega_T)},$$

where $\delta(\Omega)$ denotes the diameter of Ω and C is a constant depending on λ and the dimension n. In this paper we will observe that a more careful analysis of the argument given by Kai-Sing Tso in [5] shows that when the above conditions are satisfied and the function $f(x,t)$ does not depend on time, that is, when $f(x,t) = \phi(x)$ then

$$(3)\qquad \sup_{\Omega_T} |u| \leq C\delta(\Omega)\|\phi\|_{L^n(\Omega)},$$

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and where the constant C has the same dependence as before. We observe that the classical Pucci-Aleksandrov inequality for solutions to elliptic operators can be recovered from the above inequality (see [1, 2, 7]). For if $S = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}$ is an elliptic operator satisfying $a_{ij}(x) = a_{ji}(x)$ for all x in Ω and $i, j = 1, \ldots, n$, and where there exists $\lambda > 0$ so that

$$\lambda^{-1}|\xi|^2 \geq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbf{R}^n,$$

and $v \in C(\overline{\Omega})$ is smooth in Ω with $v = 0$ on the boundary of Ω and $|Sv(x)| \leq h(x)$ in Ω, the function $u(x, t) = v(x)t$ satisfies $|Lu(x)| \leq Th(x) + |v(x)|$, where L is the parabolic operator given by $L = S - D_t$. Hence, from (3) we get

$$\sup_{\Omega} |v| \leq C\delta(\Omega)\|h\|_{L^n(\Omega)} + T^{-1}\delta(\Omega)|\Omega|^{1/n} \sup_{\Omega} |v|,$$

where $|\Omega|$ denotes the n-dimensional Lebesgue measure of Ω. After letting T tend to infinity we obtain the maximum of v over Ω controlled by the L^n norm of h in Ω.

Proof of the inequality

Theorem. *Let $u \in C(\overline{\Omega}_T)$ and smooth in Ω_T with $u = 0$ on $\partial_p\Omega_T$ and $|Lu| \leq \phi(x)$ on Ω_T for some ϕ lying in $L^n(\Omega)$, and where L is a parabolic operator satisfying* (1). *Then, there is a constant C depending only on λ and n so that* (3) *holds.*

Proof. We may assume that for some x_0 in Ω and $0 < \tau \leq T$, $u(x_0, \tau) = \sup_{\Omega_T} |u|$. From the argument in the proof of Proposition 2.1 in [5]

$$u(x_0, \tau)^{n+1} \leq C\delta(\Omega)^n \iint_{A_u} |\det(D_{ij}u(x, t))D_t u(x, t)|\, dx\, dt,$$

where $A_u = \{(x, t) \in \Omega \times [0, \tau) :$ there exists ξ in $\mathbf{R}^n$ so that $u(y, s) \leq u(x, t) + \xi(y - x)$ for all $y \in \Omega$ and $0 \leq s \leq t\}$ and C depends only on dimension n. It is shown in [5] that at points (x, t) lying in A_u the matrix $(D_{ij}u(x, t))$ is nonpositive and $D_t u(x, t)$ is nonnegative. Since $|Lu| \leq \phi$ on Ω_T we have $-\sum_{i,j=1}^{n} a_{ij}(x, t)D_{ij}u(x, t) \leq \phi(x)$ on A_u. On the other hand, the symmetry of the coefficient matrix $(a_{ij}(x, t))$ and the nonpositiveness of the Hessian of u imply

$$\begin{aligned} |\det(D_{ij}u(x, t))| &= -\det(D_{ij}u(x, t)) \\ &\leq \lambda^{-n} \left(-\sum_{i,j=1}^{n} a_{ij}(x, t)D_{ij}u(x, t) \right)^n \leq \phi(x)^n \end{aligned}$$

for all (x, t) in A_u. Hence,

$$u(x_0, \tau)^{n+1} \leq C\lambda^{-n}\delta(\Omega)^n \iint_{A_u} \phi(x)^n D_t u(x, t)\, dx\, dt.$$

If H denotes the projection of A_u into Ω and $I(x) = \{t \in [0, \tau] : (x, t) \in A_u\}$ we have

$$u(x_0, \tau)^{n+1} \leq C\lambda^{-n}\delta(\Omega)^n \int_H \phi(x)^n \int_{I(x)} D_t u(x, t)\, dt\, dx. \tag{4}$$

Observe that if $(x, t) \in A_u$ and ξ is as in the definition of the set A_u we have $|\xi| \le u(x, t)/d(x, \partial\Omega)$. To see this let $y = x - \sigma\xi/|\xi|$, where $\sigma \ge d(x, \partial\Omega)$ is a positive number so that y lies in the boundary of Ω. Since $u(y, t) = 0$ it follows from the definition of A_u that $0 \le u(x, t) - \sigma|\xi|$. From this fact it follows that the set $I(x)$ is closed relative to $[0, \tau]$ for each x in H. Therefore, $[0, \tau]\backslash I(x)$ can be written as a disjoint union of intervals $\{I_j | j \ge 1\}$ and where each I_j is an interval having one of the following forms: $[0, a)$, (b, c), or $(d, \tau]$ with $a \le b < c \le \tau$. From the definition of $I(x)$ and when I_j has the form of one of the two first intervals, the integral $\int_{I_j} D_t u(x, t)\, dt$ is nonnegative. Among the intervals I_j there is at most one of them of the form $(d, \tau]$. Hence,

$$u(x, \tau) = \int_0^\tau D_t u(x, t)\, dt \ge \int_{I(x)} D_t u(x, t)\, dt + u(x, \tau) - u(x, d).$$

The above argument shows that for all x in H we have

$$\int_{I(x)} D_t u(x, t)\, dt \le u(x_0, \tau),$$

which together with (4) implies the theorem.

Finally, I would like to remark some new results that are consequences of the above theorem. In [8, Theorem 3.2] Ural′zeva and Ladyzenskaya show that when Ω is a C^2 domain and u denotes the solution to

$$\begin{cases} Lu = f(x, t) + \phi(x) & \text{on } \Omega_T, \\ u = 0 & \text{on } \partial_p\Omega_T, \end{cases}$$

then there are constants $C > 0$ and $\alpha \in (0, 1)$ with C and α depending on λ, n, T, p, and the C^2 character of the boundary of Ω so that

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(S_T)} + \left\| \frac{\partial u}{\partial n} \right\|_{C^{\alpha, \alpha/2}(S_T)} \le C\{\|f\|_{L^p(\Omega_T)} + \|\phi\|_{L^{n+1}(\Omega)}\}$$

for all $p > n + 2$, where $S_T = \partial\Omega \times (0, T)$ and $\partial u/\partial n$ denotes the normal derivative of $u(\cdot, t)$ on the boundary of Ω. By the parabolic analog of the methods used in [3] and (3) the above estimate can be improved in the following way: there exists C and α depending on λ, n, T, p, q, and the C^2 character of the boundary of Ω so that

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(S_T)} + \left\| \frac{\partial u}{\partial n} \right\|_{C^{\alpha, \alpha/2}(S_T)} \le C\{\|f\|_{L^p(\Omega_T)} + \|\phi\|_{L^q(\Omega)}\}$$

for all $p > n + 2$ and $q > n$.

Another interesting consequence of (3) is that if $g(x, t, y, s)$ denotes the Green's function for the operator L in the cylinder $\Omega \times \mathbf{R}$, the function $w(x, t, \cdot)$ given by

$$w(x, t, y) = \int_{-\infty}^{t} g(x, t, y, s)\, ds$$

lies in the class of weights $B_{n/n-1}(\Omega)$ independently of $(x, t) \in \Omega \times \mathbf{R}$. That is, there exists a constant C depending only on λ and n so that for all cubes Q_r of diameter r, with sides parallel to the coordinate axes and whose doubled concentric cube Q_{2r} is contained in Ω the following holds:

$$\left\{ r^{-n} \int_{Q_r} w(x, t, y)^{n/(n-1)}\, dy \right\}^{(n-1)/n} \leq C r^{-n} \int_{Q_r} w(x, t, y)\, dy.$$

This last claim can be proved by the parabolic analog of the methods used in [4, Lemma 2.0, Theorems 2.1, 2.2] to show that the Green's function $G(x, \cdot)$ on Ω of an elliptic operator S as before lies in $B_{n/n-1}(\Omega)$.

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