PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 4, AUGUST 1992

ON A REGULARITY THEOREM FOR WEAK SOLUTIONS TO TRANSMISSION PROBLEMS WITH INTERNAL LIPSCHITZ BOUNDARIES

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(Communicated by Barbara L. Keyfitz)

ABSTRACT. We show that if u is a weak solution to $\operatorname{div}(A\nabla u) = 0$ on an open set Ω containing a Lipschitz domain D, where $A = kI\chi_D + I\chi_{\Omega/D}$ $(k > 0$, $k \neq 1)$. Then, the nontangential maximal function of the gradient of u lies in $L^2(\partial D)$.

1. INTRODUCTION

This paper answers a question that was posed to us by A. Friedman in relation with a better regularity estimate for solutions to certain divergence form equations, which is necessary to prove some theorems of continuous dependence associated to some inverse problems [8].

In particular, we consider the operator $Lu = \operatorname{div}(A(X)\nabla u(X)) = 0$, where $A(X) = kI\chi_D(X) + I\chi_{\Omega/D}(X)$ $(k > 0,\ k \neq 1)$, I denotes the identity matrix, χ is the characteristic function of a set, Ω is an open set in $\mathbf{R}^n$, and D denotes a Lipschitz domain contained in Ω (see the body of the paper for the relevant definitions). If $u \in W^{1,2}(\Omega)$, the space of square integrable functions in Ω with distributional derivatives in $L^2(\Omega)$, is a weak solution to $Lu = 0$ in Ω and the gradient of u has a restriction to the boundary of D, ∂D; then the following relation should hold,

$$k\langle \nabla u^+(P), N(P)\rangle - \langle \nabla u^-(P), N(P)\rangle = 0 \quad \text{for } P \text{ on } \partial D,$$

where ∇u^+, ∇u^- denote the restrictions of the gradient of u to ∂D from the interior and exterior of D respectively, $N(P)$ denotes the exterior unit normal to ∂D, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$.

In this work we show that under the above conditions, the gradient of u actually has a restriction to ∂D and that the nontangential maximal function of the gradient of u is square integrable on ∂D. The main tools used in the proof of this theorem are the invertibility on $L^2(\partial D)$ of the operator $\lambda I - K^*$,

Received by the editors September 4, 1990 and, in revised form, February 2, 1991.

1991 *Mathematics Subject Classification*. Primary 35B65; Secondary 35C15, 35P15, 42B20, 45C05, 45E05.

The first author was partially supported by NSF grant DMS 8421377-03, and the second author was partially supported by NSF grant DMS 8915413.

where

$$K^*(f)(P) = \text{p.v.}\frac{1}{\omega_n}\int_{\partial D} \frac{\langle P-Q, N(P)\rangle}{|P-Q|^n} f(Q)\,d\sigma(Q),$$

$d\sigma$ denotes surface measure on ∂D and λ is a real number with $|\lambda| > \frac{1}{2}$, together with a natural representation of u in a neighborhood of $\overline{D}$ as the sum of a Newtonian potential and a Single Layer potential.

2. Preliminaries

The letters X, Y will denote points in $\mathbf{R}^n$, and the letters P, Q will denote points of the boundary of a domain D in $\mathbf{R}^n$. An open ball of radius r centered at the origin will be denoted as $B_r(0)$.

Definitions. A bounded domain D contained in $\mathbf{R}^n$ is called a Lipschitz domain if corresponding to each $P \in \partial D$ there is an open, right circular, double truncated cylinder $Z(P, r)$ centered at P, with radius equal to r, whose basis is at positive distance from ∂D, such that there is a rectangular coordinate system for $\mathbf{R}^n$, (x, s), $x \in \mathbf{R}^{n-1}$, $s \in \mathbf{R}$, with s-axis containing the axis of Z, and a Lipschitz function $\varphi\colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that $Z \cap D = \{(x, s) : s > \varphi(x)\} \cap Z$, and $P = (0, \varphi(0))$.

By a cone we mean an open, circular, nonempty truncated cone. Assigning one cone $\Gamma(P)$ to each P in ∂D, we call the resulting family $\{\Gamma(P) : P \in \partial D\}$ regular if there is a finite covering of ∂D by cylinders as in the above definition, such that for each $(Z(P, r), \varphi)$ there are three cones α, β, and γ each with vertex at the origin and axis along the axis of Z such that $\alpha \subsetneqq \beta/\{0\} \subsetneqq \gamma$ and for all $(x, \varphi(x)) = P \in \frac{4}{5}Z \cap \partial D$,

$$\alpha + P \subset \Gamma(P) \subset \beta + P, \qquad \gamma + P \subset Z,$$

and such that $\{\frac{4}{5}Z\}$ still covers ∂D.

We will need to approximate a given Lipschitz domain D, by sequences of C^∞ domains Ω_j, $j = 1, 2, \ldots$, in the manner described in the following lemma [6, 7].

Lemma 1. *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Then, the following propositions hold*:

(i) *There is a regular family of cones $\{\Gamma\}$ as described above.*

(ii) *There is a sequence of C^∞ domains, $\Omega_j \subset D$, and homeomorphisms $\Lambda_j\colon \partial D \to \partial\Omega_j$, such that $\sup_{P\in\partial D} |\Lambda_j(P) - P| \to 0$ as $j \to \infty$ and for all j and all $P \in \partial D$, $\Lambda_j(P) \in \Gamma(P)$.*

(iii) *There are positive functions $\omega_j\colon \partial D \to \mathbf{R}^+$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial D$,*

$$\int_E \omega_j\, d\sigma = \int_{\Lambda_j(E)} d\sigma_j$$

and $\omega_j(P)$ converges pointwise to 1 for a.e. P on ∂D. Here, $d\sigma$ and $d\sigma_j$ are the surface measures of ∂D and $\partial\Omega_j$ respectively.

(iv) *The normal vectors to Ω_j, $N(\Lambda_j(P))$, converge pointwise a.e. to $N(P)$.*

(v) *There exists a C^∞ vector field α in $\mathbf{R}^n$ such that for all j and $P \in \partial D \langle N(\Lambda_j(P)), \alpha(\Lambda_j(P))\rangle \geq C$, where C depends only on the Lipschitz character of D.*

The same scheme can be carried out but with C^∞ domains containing D.

Definition. Given a function u in D and a regular family of cones $\{\Gamma\}$, we define the nontangential maximal function of u as $u^*(P) = \operatorname{Sup}\{|u(X)| : X \in \Gamma(P)\}$ for P on ∂D.

We can define a regular family of cones and nontangential maximal functions in D^c in a similar manner.

We say that u converges nontangentially a.e. to f if for any regular family of cones $\{\Gamma\}$, we have $\lim_{x\to P\ X\in\Gamma(P)} u(X) = f(P)$ for a.e. P on ∂D.

Given a Lipschitz domain D in $\mathbf{R}^n$, we will denote the Single Layer potential of a function $f \in L^2(\partial D)$ as $\mathscr{S}(f)$, where

$$\mathscr{S}(f)(X) = \frac{1}{\omega_n(2-n)}\int_{\partial D} |X-Q|^{2-n} f(Q)\,d\sigma, \qquad X \in \mathbf{R}^n,\ n \geq 3,$$

$$\mathscr{S}(f)(X) = \frac{1}{2\pi}\int_{\partial D} \log|X-Q| f(Q)\,d\sigma, \qquad n = 2,$$

ω_n = surface measure of the unit ball in $\mathbf{R}^n$. The following facts are well known.

(i) $\mathscr{S}(f)$ is harmonic on $\mathbf{R}^n/\partial D$ and converges nontangentially to $\mathscr{S}(f) = \mathscr{S}(f)|_{\partial D}$.

(ii) The gradients of $\mathscr{S}(f)^+$ and $\mathscr{S}(f)^-$ have nontangential limits on ∂D when approaching the boundary of D from the interior and exterior respectively [2]. In particular,

$$D_i\mathscr{S}(f)^+(P) = -\tfrac{1}{2}N^i(P)f(P) + K_i(f)(P),$$

$$D_i\mathscr{S}(f)^-(P) = +\tfrac{1}{2}N^i(P)f(P) + K_i(f)(P),$$

where $N(P) = (N^1(P), \dots, N^n(P))$ and K_i denotes the following principal value operator,

$$k_i(f)(P) = \text{p.v.}\,\frac{1}{\omega_n}\int_{\partial D} \frac{P^i - Q^i}{|P-Q|^n} f(Q)\,d\sigma(Q).$$

In particular,

$$\langle \nabla\mathscr{S}(f)^{\pm}(P), N(P)\rangle = \mp\tfrac{1}{2}f(P) + K^*(f)(P), \tag{1.1}$$

where

$$K^*(f)(P) = \text{p.v.}\,\frac{1}{\omega_n}\int_{\partial D} \frac{\langle P-Q, N(P)\rangle}{|P-Q|^n} f(Q)\,d\sigma(Q). \tag{1.2}$$

(1.3) (iii) $\|K_i(f)\| \leq C\|f\|$, where C depends on the Lipschitz character of D [1], and $\|\cdot\|$ denote the norm in $L^2(\partial D)$.

(1.4) (iv) $\|(\nabla\mathcal{S}(f))^*\| + \|\mathcal{S}(f)^*\| \leq C\|f\|$, where C depends on the Lipschitz character of D and the regular family of cones [1, 2, 7].

At this point we have all the necessary ingredients to state the main result of this paper.

Theorem 1. *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ $(n \geq 2)$, which is contained in $B_{1/2}(0)$. Assume that $u \in W^{1,2}(B_1(0))$ is a weak solution to $\operatorname{div}(A\nabla u) = 0$ in $B_1(0)$, where $A(X) = kI\chi_D + I\chi_{\mathbf{R}^n/D}$, $k > 0$, $k \neq 1$. Then, the gradient of u has nontangential limits a.e. on ∂D when the boundary is approached from either side and the following estimate holds,*

$$\|(\nabla u)^*\| \leq C\|u\|_{W^{1,2}(B_1(0))}, \tag{1.5}$$

where C depends on k and the Lipschitz character of D and the chosen regular family of cones.

To prove this theorem we will use the following result.

Theorem 2. *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ $(n \geq 2)$ and K^* the operator defined in* (1.2). *Then for any real number λ with $|\lambda| > \frac{1}{2}$, the operator $\lambda I - K^*$ is invertible on $L^2(\partial D)$.*

2. Proof of Theorem 1

Let Ψ be a cut-off function supported in $B_1(0)$, $\Psi = 1$ on $B_{3/4}(0)$ and $\Psi = 0$ for $|X| > \frac{7}{8}$. From the hypothesis, it follows that $u\Psi \in W^{1,2}(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n/\partial D)$ and $h = \Delta(u\Psi) = u\Delta\Psi + 2\nabla u\nabla\Psi$ on $\mathbf{R}^n/\partial D$. In particular, we observe that h is square integrable on $\mathbf{R}^n$ with compact support contained in $B_1(0)$ and away from $\overline{D}$. Let v denote the Newtonian Potential of h,

$$v(X) = \frac{1}{\omega_n(2-n)}\int_{\mathbf{R}^n} |X-Y|^{2-n} h(Y)\,dY, \qquad X \in \mathbf{R}^n,\ n \geq 3,$$

$$v(X) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \log|X-Y| h(Y)\,dY, \qquad n = 2.$$

For any $f \in L^2(\partial D)$, the function $w = u\Psi + v - \mathcal{S}(f)$ is harmonic on $\mathbf{R}^n/\partial D$ and lies in $W^{1,2}_{\text{loc}}(\mathbf{R}^n)$. We claim that Theorem 2 implies that one might choose f so that w becomes a weak solution to $\operatorname{div}(A\nabla u) = 0$ on the whole $\mathbf{R}^n$. With this choice of f, the maximum principle for weak solutions to divergence form operators implies that

$$\sup_{B_R(0)} |w| \leq \sup_{\partial B_R(0)} |w|,$$

but for $n \geq 3$, $w = O(|X|^{2-n})$ at infinity. Then w would be identically zero.

Therefore, it only remains to prove the claim. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ be a test function, and Ω_j^+, Ω_j^- denote two sequences of C^∞ domains as in Lemma 1 such that $\Omega_j^+ \subset D \subset \Omega_j^-$; then

$$\begin{aligned}
\int_{\mathbf{R}^n} A(X)\nabla w \nabla \phi\, dX &= \lim_{j\to\infty}\left(\int_{\Omega_j^+} k\nabla w\nabla\phi\, dX + \int_{\mathbf{R}^n/\Omega_j^-} \nabla w \nabla \phi\, dX\right)\\
&= \lim_{j\to\infty}\left(\int_{\partial\Omega_j^+} k\langle\nabla w\,,\, N_j^+\rangle\phi\, d\sigma_j^+ - \int_{\partial\Omega_j^-}\langle\nabla w\,,\, N_j^-\rangle\phi\, d\sigma_j^-\right)\\
&= \lim_{j\to\infty}\left(\int_{\partial\Omega_j^+} k\langle\nabla u\,,\, N_j^+\rangle\phi\Psi\, d\sigma_j^+ - \int_{\partial\Omega_j^-}\langle\nabla u\,,\, N_j^-\rangle\phi\Psi\, d\sigma_j^-\right)\\
&\quad + \lim_{j\to\infty}\int_{\partial\Omega_j^+} k\{\langle\nabla v\,,\, N_j^+\rangle - \langle\nabla\mathscr{S}(f)\,,\, N_j^+\rangle\}\phi\, d\sigma_j^+\\
&\quad - \int_{\partial\Omega_j^-}\{\langle\nabla v\,,\, N_j^-\rangle - \langle\nabla\mathscr{S}(f)\,,\, N_j^-\rangle\}\phi\, d\sigma_j^-.
\end{aligned}$$

The first term in the last sum equals

$$\begin{aligned}
&\lim_{j\to\infty}\left(\int_{\Omega_j^+} k\nabla u\nabla(\phi\Psi)\, dX + \int_{B_1(0)/\Omega_j^-}\nabla u\nabla(\phi\Psi)\, dX\right)\\
&\quad = \int_{B_1(0)} A(X)\nabla u\nabla(\phi\Psi)\, dX = 0.
\end{aligned}$$

From (1.1), Lemma 1, the L^2 estimate for the nontangential maximal function of the gradient of $\mathscr{S}(f)$ (1.4), and the fact that v is harmonic in a neighborhood of $\overline{D}$, the second limit above is equal to

$$\int_{\partial D}\left\{(k-1)\langle\nabla v\,,\, N\rangle - \frac{k+1}{2}f + (k-1)K^*(f)\right\}\phi\, d\sigma.$$

Therefore, if $f \in L^2(\partial D)$ solves the integral equation $(\frac{k+1}{2(k-1)}I - K^*)f = \langle\nabla v\,,\, N\rangle$ on ∂D then the corresponding w satisfies $\operatorname{div}(A\nabla w) = 0$ on $\mathbf{R}^n$, which proves the claim.

The estimate (1.5) is an easy consequence of (1.4), Theorem 2, and the following estimate,

$$\|\nabla v\|_{L^\infty(B_{1/2}(0))} \le C\|u\|_{W^{1,2}(B_1(0))}.$$

Remark. When $n = 2$ the same argument goes through after one observes the following facts;

(i) The function h has mean value zero on $\mathbf{R}^n$.

$$\begin{aligned}
\int_{\mathbf{R}^n} h(Y)\, dY &= \int_{B_1(0)/D}\{\operatorname{div}(u\nabla\Psi) + \nabla u\nabla\Psi\}\, dY\\
&= \int_{B_1(0)/D}\nabla u\nabla\Psi\, dY = \int_{B_1(0)} A\nabla u\nabla\Psi\, dY = 0.
\end{aligned}$$

(ii) Since v is harmonic on a neighborhood of $\overline{D}$, the function $\langle\nabla v\,,\, N\rangle$ has mean value zero on ∂D. This, together with the fact that $K(1) = \frac{1}{2}$, where K denotes the adjoint operator to K^* on ∂D [2], implies that the solution f

to the above integral equation also has mean value zero on ∂D. Therefore, one can show that $w = O(|X|^{-1})$ at infinity. This finishes the proof of Theorem 1.

3. Proof of Theorem 2

Let λ be a real number with $|\lambda| > \frac{1}{2}$ and set $Tf = (\lambda I - K^*)(f)$ for f in $L^2(\partial D)$. It was already proved by O. D. Kellogg (Foundations of Potential Theory), that the eigenvalues of K^* on $L^2(\partial D)$ lie in $(-\frac{1}{2}, \frac{1}{2}]$ for smooth domains; but his argument also goes through for Lipschitz domains; we will include it here for the sake of completeness.

(i) *T is one to one on $L^2(\partial D)$.* The argument is by contradiction. Suppose that $f \in L^2(\partial D)$ satisfies $Tf = 0$ and f is not identically zero. Since $K(1) = \frac{1}{2}$, it follows by duality that f has mean value zero on ∂D. Hence, $\mathcal{S}(f) = 0(|X|^{1-n})$ and $|\nabla \mathcal{S}(f)| = O(|X|^{-n})$ at infinity for $n \geq 2$. Since f is not identically zero, the following numbers cannot be zero,

$$A = \int_D |\nabla \mathcal{S}(f)|^2 \, dX \quad \text{and} \quad B = \int_{\mathbf{R}^n/D} |\nabla S(f)|^2 \, dX.$$

On the other hand, since $|\nabla \mathcal{S}(f)|^2 = \frac{1}{2}\Delta(\mathcal{S}(f)^2)$, we have

$$A = \int_{\partial D} \left(-\frac{1}{2} f + K^*\right)(f) \mathcal{S}(f) \, d\sigma \quad \text{and} \quad B = -\int_{\partial D} \left(\frac{1}{2} f + K^*\right)(f) \mathcal{S}(f) \, d\sigma.$$

Since $T(f) = 0$, it follows that $\lambda = \frac{1}{2}\frac{B-A}{B+A}$. Thus $|\lambda| \leq \frac{1}{2}$, which is a contradiction.

Lemma 2. *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ and α denote a smooth vector field on $\mathbf{R}^n$ as in Lemma* 1(v). *Then, there exists a constant C depending on λ, n, and the Lipschitz character of D, such that for all f in $L^2(\partial D)$ the following estimate holds,*

$$\|f\| \leq C\{\|(\lambda I - K^*)f\| + \|S(f)\| + \|H(f)\|\}, \tag{1.6}$$

where H denotes the following operator on $L^2(\partial D)$,

$$H(f)(P) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P - Q, \alpha(P) - \alpha(Q)\rangle}{|P - Q|^n} f(Q) \, d\sigma(Q).$$

Proof. In what follows, C denotes a constant depending on the Lipschitz character of D and λ. Let $f \in L^2(\partial D)$ and set $u(X) = \mathcal{S}(f)(X)$. From the Rellich-identities [3, 4], we have that

$$\operatorname{div}(\tfrac{1}{2}\alpha|\nabla u|^2) = \operatorname{div}(\langle \alpha, \nabla u\rangle \nabla u) + \tfrac{1}{2}|\nabla u|^2 \operatorname{div} \alpha - D_i \alpha_j D_i u D_j u.$$

After integrating this identity on D, applying Green's formula, and observing that

$$|\langle \nabla u, N\rangle| = |(\lambda - \tfrac{1}{2})f - T(f)| \leq |\nabla u|,$$
$$\langle \alpha, \nabla u\rangle = -\tfrac{1}{2}\langle \alpha, N\rangle f + K_\alpha(f) \quad \text{on } \partial D,$$

where

$$K_\alpha(f)(P) = \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P - Q, \alpha(P)\rangle}{|P - Q|^n} f(Q) \, d\sigma(Q)$$

and

$$\int_D |\nabla u|^2 \, dX = \int_{\partial D} S(f)[(\lambda -; \tfrac{1}{2})f - T(f)] \, d\sigma,$$

we get

$$\frac{1}{2}\int_{\partial D}\langle\alpha, N\rangle\left(\lambda-\frac{1}{2}\right)^2 f^2\,d\sigma$$
$$\leq \int_{\partial D}\left[-\frac{1}{2}\langle\alpha, N\rangle f + K_\alpha(f)\right]\left[\left(\lambda-\frac{1}{2}\right)f - T(f)\right]d\sigma$$
$$+ C\|f\|\{\|S(f)\| + \|T(f)\|\} + C\|S(f)\|\,\|T(f)\|.$$

Multiplying out the integrand in the second integral above and taking to the left-hand side of the inequality the term involving f^2, we obtain

$$\frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)\int_{\partial D}\langle\alpha, N\rangle f^2\,d\sigma$$
$$\leq \left(\lambda-\frac{1}{2}\right)\int_{\partial D}K_\alpha(f)f\,d\sigma + C\|f\|\{\|S(f)\| + \|T(f)\|\} + C\|S(f)\|\,\|T(f)\|.$$

If K_α^* denotes the adjoint operator on $L^2(\partial D)$ of the operator K_α, it is easy to observe that $K_\alpha^* + K_\alpha = H$, where H is the operator defined in Lemma 2 and by duality

$$\int_{\partial D}K_\alpha(f)f\,d\sigma = \frac{1}{2}\int_{\partial D}H(f)f\,d\sigma.$$

Since $|\lambda| > \frac{1}{2}$, the $\|f\|$ can be hidden on the left-hand side of the above inequality, which proves Lemma 2.

Since the operators S and H are compact on $L^2(\partial D)$, we conclude from Lemma 2 that $\lambda I - K^*$ has closed range.

(ii) If λ is real and $|\lambda| > \frac{1}{2}$ then $\lambda I - K^*$ is surjective on $L^2(\partial D)$ and hence is invertible on $L^2(\partial D)$.

Suppose to the contrary that for some λ real, $|\lambda| > \frac{1}{2}$, $\lambda I - K^*$ is not invertible on $L^2(\partial D)$. Then the intersection of the spectrum of K^* and the set $\{\lambda \in R : |\lambda| > \frac{1}{2}\}$ is not empty and so there exists a real number λ that belongs to this intersection and is a boundary point of the set. To reach a contradiction we will show that $\lambda I - K^*$ is invertible.

From (i) and Lemma 2 we know that $\lambda I - K^*$ is injective and has closed range. Hence there is a constant C such that for all $f \in L^2(\partial D)$.

$$(*) \qquad \|f\|_{L^2(\partial D)} \leq C\|(\lambda I - K^*)f\|_{L^2(\partial D)}.$$

Also since λ is a boundary point of the spectrum of K^* and the real line there exists, a sequence of real numbers λ_j with $|\lambda_j| > \frac{1}{2}$, $\lambda_j \to \lambda$, and $\lambda_j I - K^*$ is invertible on $L^2(\partial D)$. In particular given $g \in L^2(\partial D)$ there exists unique $f_j \in L^2(\partial D)$ such that $(\lambda I - K^*)f_j = g$.

If $\{\|f_j\|\}$ has a bounded subsequence then there exists another subsequence that converges weakly to some f in $L^2(\partial D)$. Setting $T = \lambda I - K^*$, we have

$$\int_{\partial D}T(f)h\,d\sigma = \lim_{j\to\infty}\int_{\partial D}f_j T^*(h)\,d\sigma = \lim_{j\to\infty}\int_{\partial D}T(f_j)h\,d\sigma = \int_{\partial D}gh\,d\sigma.$$

Hence $Tf = g$.

In the opposite case we may assume $\|f_j\| = 1$ and $(\lambda I - K^*)(f_j)$ converges to zero in $L^2(\partial D)$. However from $(*)$

$$1 = \|f_j\| \leq C\|(\lambda I - K^*)f_j\| \leq C\|(\lambda_j I - K^*)f_j\| + C|\lambda - \lambda_j|.$$

Since the final two terms converge to zero as $j \to \infty$, we arrive at a contradiction. We conclude that for each λ real, $|\lambda| > \frac{1}{2}$, $\lambda I - K^*$ is invertible.

The above argument for (ii) is due to Mark Sand, and we thank him for allowing us to use this simplification of our previous proof.

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