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LIPSCOMB'S $L(A)$ SPACE FRACTALIZED IN HILBERT'S $l^2(A)$ SPACE

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ABSTRACT. By extending adjacent-endpoint identification in Cantor's space $N(\{0, 1\})$ to Baire's space $N(A)$, we move from the unit interval $I = L(\{0, 1\})$ to $L(A)$. The metric spaces $L(A)^{n+1}$ and $L(A)^\infty$ have provided nonseparable analogues of Nöbeling's and Urysohn's imbedding theorems. To date, however, $L(A)$ has no metric description. Here, we imbed $L(A)$ in $l^2(A)$ and the induced metric yields a geometrical interpretation of $L(A)$. Except for the small last section, we are concerned with the imbedding. Once inside $l^2(A)$, we see $L(A)$ as a subspace of a "closed simplex" Δ^A having the standard basis vectors together with the origin as vertices. The part of $L(A)$ in each n-dimensional face σ^n of Δ^A is a "generalized Sierpiński Triangle" called an *n-web* ω^n. Topologically, ω^n is $L(\{0, 1, \dots, n\})$. For $n = 2$, ω^2 is just the usual Sierpiński Triangle in E^2; for $n = 3$, ω^3 is Mandelbrot's fractal skewed web. Thus, $L(A) \to l^2(A)$ invites an extension of fractals. That is, when $|A|$ infinite, Baire's Space $N(A)$ is a "generalized code space" on $|A|$ symbols that addresses the points of the "generalized fractal" $L(A)$.

1. NOTATION AND DEFINITIONS

For any set A, let $N(A)$ be the set of all sequences of elements of A. For $\boldsymbol{a} = a_1a_2\cdots$ and $\boldsymbol{b} = b_1b_2\cdots$ in $N(A)$, define

$$\rho(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{\min\{k|a_k \neq b_k\}}.$$

Then $N(A)$ with this metric ρ is a *generalized Baire's* 0*-dimensional space.* Topologically $N(A)$ is the countable product of discrete space A [11, p. 51]. In particular, when $A = \{0, 1\}$ we have the mappings

$$(1) \qquad a_1a_2\cdots \leftrightarrow \sum_{i=1}^{\infty} \frac{2a_i}{3^i} \to \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

That is, there is the topological correspondence ("$\leftrightarrow$" in (1)) between $N(A)$ and Cantor's middle-thirds space $\mathscr{C}(0, 1)$; also, there is an identification map

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("$\rightarrow$" in (1)) onto the unit interval I—the so-called *identification of adjacent endpoints.*

The equivalence, "$\leftrightarrow$" in (1), reveals the *endpoints of Cantor's space* as eventually constant sequences of 0's and 1's. Generalizing this observation to arbitrary A, *endpoints of* $N(A)$ are defined [4] as the eventually constant sequences in A. If an endpoint is a constant sequence, then its *tail-index* is defined as zero. Otherwise, the *tail-index of an endpoint* $a_1a_2\cdots$ is the unique index j such that $a_j \neq a_{j+1} = a_{j+2} = a_{j+3} = \cdots$. Two distinct endpoints $\boldsymbol{a}$ and $\boldsymbol{b}$ of $N(A)$ are *adjacent* when $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same tail-index $j \geq 1$; $a_k = b_k$ for each $k < j$; $a_j = b_{j+1} = b_{j+2} = \cdots$; and $b_j = a_{j+1} = a_{j+2} = \cdots$. A string of repeated a's is denoted $\overline{a}$, e.g., $a_1a_2aaa\cdots = a_1a_2\overline{a}$.

For $\boldsymbol{a}, \boldsymbol{b} \in N(A)$ define $\boldsymbol{a} \sim \boldsymbol{b}$ when either $\boldsymbol{a} = \boldsymbol{b}$ or $\boldsymbol{a} \neq \boldsymbol{b}$ are adjacent endpoints. An induced equivalence class α is either a singleton or a doubleton subset of $N(A)$. Any member $\boldsymbol{a} = a_1a_2\cdots$ of α is an *expansion of* α. Thus, each α has at most two expansions. The space $L(A)$ is the quotient space $N(A)/\sim$ and the natural map $p\colon N(A) \rightarrow L(A)$ is perfect [4], i.e., p is a continuous closed surjection with compact point-inverse sets. It follows that $L(A)$ is metric [2, p. 235; 9; 15] and that $L(A)$ is one-dimensional [4] (the dimension is the *covering dimension* [11]). For applications of $L(A)$, [5–7, 12] contains nonseparable versions of Nöbeling's and Urysohn's imbedding theorems [14, 16].

A doubleton set $\alpha \in L(A)$ is a *rational* point in $L(A)$. All other points of $L(A)$ are *irrational*. The p-image of any constant sequence is an *endpoint* of $L(A)$. The points of $L(A)$ can also be partitioned [7] into those of *finite character* and those of *infinite* character; That is, α has finite character only if some expansion $a_1a_2\cdots$ of α has only finitely many symbols—$\{a_1, a_2, \ldots\}$ is finite. Every rational point of $L(A)$ has finite character while irrational points appear in both varieties.

Let $\bigcup\{I(a)|a \in A\}$ be a system of unit segments $[0, 1]$. By identifying all zeros in $\bigcup\{I(a)|a \in A\}$ we get a star-shaped set $S(A)$. Defining a metric d_S in $S(A)$ by

$$d_S(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same segment } I(a), \\ x + y & \text{if } x, y \text{ belong to distinct segments}, \end{cases}$$

we obtain a metric space called *star-shaped with index* A. The space $S(A)$ was used by Nagata [13] and Kowalsky [3] to construct general imbedding theorems [7].

Turning to *Hilbert's Space*, for any set A let E^A be the cartesian product of $|A|$ copies of the real line E^1. The metric space $l^2(A)$ is that having (1) *elements*: every $x = \{x_a\} \in E^A$ such that $x_a = 0$ for all but at most countably many $a \in A$ and $\sum x_a^2$ converges; and (2) *topology*: that induced by the metric $d(x, y) = \sqrt{\sum_a (x_a - y_a)^2}$.

Generalized Euclidean space $\mathscr{E}^A$ is the subspace of $l^2(A)$ consisting of those $\{x_a\}$ with $x_a = 0$ for all but at most finitely many $a \in A$. If $\mathscr{V}(A)$ denotes the set containing all the standard basis vectors $\mathbf{u}_a$, $a \in A$, and the zero vector $\mathbf{0}$, then the *infinite-dimensional simplex* Δ^A (in $l^2(A)$) is the convex hull (an intersection of closed convex sets) of $\mathscr{V}(A)$.

If $A = \{0, 1, \ldots, n\}$ then, as we shall see from the imbedding (given below) of $L(A)$ into $l^2(A)$, the (one-dimensional when $n > 0$) space $p(N(A)) = L(A)$

is a closed subspace of the n-simplex σ_0^n. In this case, the vertices of σ_0^n are $\mathbf{u}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{u}_n = (0, \ldots, 0, 1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$. For $A = \{0, 1, \ldots, n\}$ an *n-web* ω^n is either the subspace $L(A)$ of σ_0^n or any image of $L(A)$ in any n-simplex σ^n under a linear homeomorphism [10] from σ_0^n onto σ^n. Any simplicial complex K induces a *web complex*; for every n replace each σ^n in K with its subspace ω^n.

The term n-web is introduced here but was motivated by the title *A Fractal Skewed Web* of plate 143 of Mandelbrot's book [8, p. 142]. The term *web complex* is introduced to increase the understanding of the dense subspace $M = \bigcup_n M_n$ of $L(A)$ where $M_0 \subset M_1 \subset \cdots$. Indeed, each M_n is the web complex induced from the n-skeleton of the (possibly infinite-dimensional) simplex Δ^A. One can also view M_n as those points α in $L(A)$ whose expansions contain at most n symbols of A. Thus, each M_n contains both rational and irrational points of $L(A)$, and M is the set of points of $L(A)$ that have finite character.

2. Projecting $N(A)$ onto Cantor subspaces

Let $z \in A$ be fixed and define $A' = A - \{z\}$. For each element b of A', $N(A)$ has a subspace $\mathscr{C}(z, b)$ of sequences whose values lie in $\{z, b\} : \mathscr{C}(z, b)$ is a copy of Cantor's space. For each $b \in A'$ define a continuous projection $\pi_b \colon N(A) \to \mathscr{C}(z, b)$ as follows:

$$\pi_b \colon a_1 a_2 \cdots \mapsto a_1^b a_2^b \cdots \quad \text{where } a_k^b = \begin{cases} b & \text{if } b = a_k, \\ z & \text{otherwise}. \end{cases}$$

The map π_b is an open map since basic open sets

$$\langle a_1, \ldots, a_n \rangle = \{a_1\} \times \cdots \times \{a_n\} \times A \times A \times \cdots$$

are mapped to open sets

$$\pi_b(\langle a_1, \ldots, a_n \rangle) = \langle a_1^b, \ldots, a_n^b \rangle \subset \mathscr{C}(z, b).$$

The map π_b is continuous since the π_b-inverse image of $\langle x_1, \ldots, x_n \rangle \subset \mathscr{C}(z, b)$ is an open set of the form

$$\left\langle \bigtimes_J (A - \{b\})_j, \ \bigtimes_K \{b\}_k \right\rangle,$$

where $J \subset \{1, 2, \ldots, n\}$ is the set of indices j for which $x_j \neq b$ and K is the other set of indices, i.e., those for which $x_k = b$. However, when A is infinite, then $N(A)$ is not compact and the projection π_b is not a closed map: Let $\{a_1, a_2, \ldots\}$ be an infinite subset of A. Then the set

$$F = \{a_1 b \overline{a}_1, a_2 a_2 b \overline{a}_2, a_3 a_3 a_3 b \overline{a}_3, \ldots\} \tag{2}$$

is closed in $N(A)$ while $\pi_b(F)$ is not closed in $\mathscr{C}(z, b)$. These observations are summarized in the following theorem.

Theorem 1. *The projection $\pi_b \colon N(A) \to \mathscr{C}(z, b)$ is a continuous open map. Also, π_b is not closed $\Leftrightarrow A$ is infinite.*

3. Projecting $L(A)$ onto unit interval subspaces

Again, let $z \in A$ be fixed and define $A' = A - \{z\}$. Call $p(\overline{z}) = \zeta$ the *zero of* $L(A)$. Let β be any other endpoint of $L(A)$, i.e., $\beta = p(\overline{b})$ for some

$b \in A'$. Using projection π_b and identification p, we can induce a projection within $L(A)$ that takes $L(A)$ onto the subspace $p(\mathscr{C}(z, b)) = I(\zeta, \beta)$ of $L(A)$. Diagram (3) is illustrative: p_b is the restriction of p, the induced projection is π_β, and the subspace $I(\zeta, \beta)$ of $L(A)$ is a unit interval.

$$\begin{array}{ccc} N(A) & \xrightarrow{\pi_b} & \mathscr{C}(z, b) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_b} \\ L(A) & \overset{\pi_\beta}{- - \rightarrow} & I(\zeta, \beta). \end{array} \tag{3}$$

Since π_b maps adjacent endpoints in $N(A)$ onto adjacent endpoints in $\mathscr{C}(z, b)$, $p_b \circ \pi_b$ is constant on the fibers of p. Thus, π_β is well defined. Because the diagram (3) is commutative and the map p is closed, π_β pulls closed sets back to closed sets and is therefore continuous.

Lemma 2. *The projection $\pi_\beta\colon L(A) \to I(\zeta, \beta)$ is a continuous map. Also, π_β is not closed $\Leftrightarrow$ A is infinite.*

Proof. Only the characterization of π_β being not closed remains. On the one hand, if F is the closed subset of $N(A)$ defined by (2) then $p(F)$ is a closed subset of $L(A)$. It follows that

$$F' = p^{-1}p(F) = F \cup \{a_1a_1\overline{b}, a_2a_2a_2\overline{b}, a_3a_3a_3a_3\overline{b}, \dots\}$$

is closed in $N(A)$. But $p_b(\pi_b(F')) = \pi_\beta(p(F))$ is not closed in $I(\zeta, \beta)$. On the other hand, if A is finite then $N(A)$ is compact. In this case, π_b is closed, and it follows that the π_β pushes closed subsets of $L(A)$ to closed sets. □

Lemma 3. *The projection $\pi_\beta\colon L(A) \to I(\zeta, \beta)$ is not open $\Leftrightarrow$ A has at least three members.*

Proof. ($\Leftarrow$) Let $B = \langle b, z\rangle$ be a basic open set in $N(A)$ where $b \neq z$. Let E_B be the set of points $\mathbf{a}$ in B such that $a_3 = a_4 = \cdots$. Then $H = B - E_B$ is an open set in $N(A)$ and $p^{-1}p(H) = H$. Thus $p(H)$ is open in $L(A)$. However, since A has at least three members,

$$\pi_b(H) = \langle b, z\rangle - \{bz\overline{b}\}$$

and $bz\overline{z} \in \pi_b(H)$. But then $p_b \circ \pi_b(p(H))$ is the half-closed interval

$$[p_b(bz\overline{z}), p_b(bz\overline{b})).$$

($\Rightarrow$) Since $\zeta \neq \beta$ and A has two members, the projection is a homeomorphism. □

The following theorem is the analogue of Theorem 1 and is a combination of Lemmas 2, 3.

Theorem 4. *The projection $\pi_\beta\colon L(A) \to L(\zeta, \beta)$ is a continuous map. Also, π_β is not closed $\Leftrightarrow$ A is infinite. Furthermore, π_β is not open $\Leftrightarrow$ A has at least three members.*

If $\pi_\beta(\alpha) \neq \zeta$ then we say that α has a *nonzero projection into* $I(\zeta, \beta)$. The following lemma is immediate.

Lemma 5. *Each member of $L(A)$ has a nonzero projection into at most a countable number of the subspaces $I(\zeta, \beta)$.*

4. The star in $L(A)$ and $l^2(A)$

The subspace $\bigcup_\beta I(\zeta, \beta)$ of $L(A)$ is homeomorphic to a star-space $S(A)$ with index set A [7] whenever A is infinite. There is also a copy of $S(A)$ in Hilbert's $l^2(A)$ space: Let $\mathbf{u}_b$, $b \in A$, denote the unit vectors in $l^2(A)$, i.e.,

$$\mathbf{u}_b = \{x_a\} \quad \text{where } x_b = 1 \text{ and } x_a = 0 \text{ otherwise}.$$

Then for $b \in A'$ the subspace

$$[\mathbf{0}, \mathbf{u}_b] = \{t\mathbf{u}_b | 0 \le t \le 1\}$$

of $l^2(A)$ is a copy of the unit interval. Furthermore, when A is infinite, $S(A)$ is homeomorphic to the metric subspace $\bigcup_{b\in A'} [\mathbf{0}, \mathbf{u}_b]$ of $l^2(A)$.

The next lemma exhibits a homeomorphism ψ_β from the "βth arm," $I(\zeta, \beta)$ of the star in $L(A)$ to the unit interval. The induced homeomorphism $\alpha \mapsto \psi_\beta(\alpha)\mathbf{u}_b$ from the βth arm of the star in $L(A)$ to the bth arm of the star in $l^2(A)$ is fundamental to the embedding of $L(A)$ into $l^2(A)$.

Lemma 6. *If we identify z with 0 and b with 1, then we can induce a homeomorphism $\psi_b \colon \mathscr{C}(z, b) \to \mathscr{C}(0, 1)$. Then, from ψ_b and the identifications (p_b and p_1) of adjacent endpoints, we induce a unique topological correspondence ψ_β from $I(\zeta, \beta)$ onto the unit interval $[0, 1]$ that makes the following diagram commute.*

$$\begin{array}{ccc} \mathscr{C}(z, b) & \xrightarrow{\psi_b} & \mathscr{C}(0, 1) \\ {\scriptstyle p_b}\downarrow & & \downarrow{\scriptstyle p_1} \\ I(\zeta, \beta) & \overset{\psi_\beta}{\dashrightarrow} & [0, 1]. \end{array} \tag{4}$$

For each $\alpha \in L(A)$ and each $b \in A'$, define

$$\alpha^b = \psi_\beta \circ \pi_\beta(\alpha).$$

Also, for $b = z$ define $\alpha^b = 0$. Combining diagrams (3) and (4), we see $0 \le \alpha^b \le 1$ for each $b \in A$ and, from Lemma 5, $0 < \alpha^b$ for at most a countable number of $b \in A$. We can say more:

Lemma 7. *For each $\alpha \in L(A)$ we can choose a binary expansion $x_1^b x_2^b \cdots$ of each α^b such that for each subscript $i \in \{1, 2, \dots\}$ there is at most one $b \in A$ with $x_i^b = 1$. From this it follows that*

$$\sum_{b\in A} \alpha^b \le 1. \tag{5}$$

Proof. For each $\alpha \in L(A)$ choose $\phi(\alpha) = a_1 a_2 \cdots \in \alpha$. Then for $b \ne z$ use the choice function ϕ to calculate

$$\begin{aligned} \alpha^b &= \psi_\beta \circ \pi_\beta(\alpha) = p_1 \circ \psi_b \circ \pi_b(a_1 a_2 \cdots) \\ &= p_1 \circ \psi_b(a_1^b a_2^b \cdots) = \sum_{i=1}^{\infty} \frac{x_i^b}{2^i}, \end{aligned}$$

where

(6) $$x_i^b = 1 \Leftrightarrow a_i^b = b \Leftrightarrow a_i = b\,.$$

Now let $i \in \{1, 2, \dots\}$ be fixed. Then a_i is a fixed member of A. Thus, $a_i = b$ for exactly one member $b \in A$. But then, via (6), $x_i^b = 1$ for exactly one $b \in A$. To see that the inequality in (5) holds, note that for $i = 1$ there is at most one of $b \in A$ such that $x_i^b = 1$, i.e., at most one of the α^b's contribute $1/2$ to the sum in (5). An induction argument finishes the proof. □

Corollary 8. *For each* $\alpha \in L(A)$,

$$\sum_{b\in A} (\alpha^b)^2 \le 1\,.$$

5. The imbedding map

For each point $\alpha \in L(A)$, map α to the point $f(\alpha) = \{\alpha^b\} \in l^2(A)$. Corollary 8 above shows that f is into $l^2(A)$.

Lemma 9. *The map* $f\colon L(A) \to l^2(A)$ *is injective.*

Proof. Let α and γ be distinct members of $L(A)$ and suppose that $\boldsymbol{a} \in \alpha$ and $\boldsymbol{c} \in \gamma$. It follows that

(7) $$\boldsymbol{a} \neq \boldsymbol{c} \quad \text{and} \quad \boldsymbol{a} \nsim \boldsymbol{c}.$$

From the first condition in (7), we can choose j as the smallest index such that

(8) $$a = a_j \neq c_j = c\,.$$

(Without loss of generality, we assume that $a \neq z$.) Now evaluate the projection π_a at the points $\boldsymbol{a}$ and $\boldsymbol{c}$. It follows from (8) that

(9) $$\pi_a(\boldsymbol{a}) \neq \pi_a(\boldsymbol{c})\,.$$

Statement (9) coupled with the fact that ψ_a is a homeomorphism show that either $f(\alpha) \neq f(\gamma)$ (in which case we are finished) or

(10) $$\pi_a(\boldsymbol{a}) \text{ is adjacent to } \pi_a(\boldsymbol{c})\,.$$

Thus, for the rest of this proof we assume (10) to be true. Now from (8) and the definition of j, both points in (10) have tail-index j. Furthermore, (10) and $(\pi_a(\boldsymbol{a}))_j = a \neq z$, imply $a = c_{j+1} = c_{j+2} = \cdots$. If $c \neq z$ then similarly we conclude $c = a_{j+1} = a_{j+2} = \cdots$. But then $\boldsymbol{a} \sim \boldsymbol{c}$, which contradicts the second condition in (7). Thus, only the case $c = z$ remains: In this case, we have

$$\boldsymbol{c} = a_1 \cdots a_{j-1} z a a \cdots \quad \text{and} \quad \boldsymbol{a} = a_1 \cdots a_{j-1} a a_{j+1} a_{j+2} \cdots\,.$$

From this representation of $\boldsymbol{c}$, and the second condition in (7), there is an index $k > j$ such that $a_k \neq z$. If $a_k = a$ then we contradict (10). Thus, for this index k we can suppose $a \neq a_k \neq z$. Then

$$\pi_{a_k}(\boldsymbol{a}) \neq \pi_{a_k}(\boldsymbol{c}) \quad \text{and} \quad \pi_{a_k}(\boldsymbol{a}) \nsim \pi_{a_k}(\boldsymbol{c})\,.$$

Therefore, in this final case, and consequently in every case, we have $f(\alpha) \neq f(\gamma)$. □

6. The embedding map factored through $N(A)$

To see that the imbedding map f is both continuous and open, we first combine diagrams (3) and (4). That is, for each $b \in A'$, define g_b as the diagonal map.

$$\begin{array}{ccccc} N(A) & \xrightarrow{\pi_b} & \mathscr{C}(z, b) & \xrightarrow{\psi_b} & \mathscr{C}(0, 1) \\ {\scriptstyle p}\downarrow & & \searrow^{g_b} & & \downarrow{\scriptstyle p_1} \\ L(A) & \xrightarrow[\pi_\beta]{} & I(\zeta, \beta) & \xrightarrow[\psi_\beta]{} & [0, 1] \end{array} \tag{11}$$

Then define $g: N(A) \to l^2(A)$ as

$$g(\boldsymbol{a}) = \{p_1 \circ \psi_b \circ \pi_b(\boldsymbol{a})\} = \{g_b(\boldsymbol{a})\},$$

where g_z is the zero map. From the definition of f we have

$$g(\boldsymbol{a}) = f \circ p(\boldsymbol{a}) = \{\psi_\beta \circ \pi_\beta \circ p(\boldsymbol{a})\}.$$

That is, diagram (12) is commutative $(g = f \circ p)$.

$$\begin{array}{ccc} N(A) & \xrightarrow{g} & l^2(A) \\ {\scriptstyle p}\searrow & & \nearrow{\scriptstyle f} \\ & L(A) & \end{array} \tag{12}$$

Since f is injective and $g = f \circ p$, we calculate that g and p have the same fibers,

$$g^{-1}(f(\alpha)) = p^{-1}(f^{-1}(f(\alpha))) = p^{-1}(\alpha).$$

Lemma 10. *The map $g: N(A) \to l^2(A)$ is continuous.*

Proof. It suffices to show that if $\boldsymbol{a}^n \to \boldsymbol{a}$ in $N(A)$, then

$$d(g(\boldsymbol{a}^n), g(\boldsymbol{a})) \to 0.$$

Let $k > 0$ be given and define $A_k = \{a_1, \dots, a_k\}$, i.e., A_k contains the first k coordinates of $\boldsymbol{a}$, A'_k the other coordinates of $\boldsymbol{a}$, and A'_{nk} the complementary set, relative to the coordinates of $\boldsymbol{a}^n$, of the first k coordinates of $\boldsymbol{a}^n$.

Since $\boldsymbol{a}^n \to \boldsymbol{a}$, there is an N such that whenever $n > N$,

$$a_i^n = a_i \quad \text{for } 1 \le i \le k.$$

Then, when $n > N$,

$$\begin{aligned}(d(g(\boldsymbol{a}^n), g(\boldsymbol{a})))^2 &= \sum_{b \in A} |g_b(\boldsymbol{a}^n) - g_b(\boldsymbol{a})|^2 \\ &\le \sum_{b \in A_k} |\cdots| + \sum_{b \in A'_k} |\cdots| + \sum_{b \in A'_{nk}} |\cdots| \\ &\le \frac{k}{2^k} + \frac{1}{2^k} + \frac{1}{2^k} = \frac{k+2}{2^k}. \quad \square\end{aligned}$$

Lemma 11. *The map $f: L(A) \to l^2(A)$ is continuous.*

Proof. Since $g: N(A) \to l^2(A)$ is continuous, g pulls back open sets in $l^2(A)$ to open g-inverse sets in $N(A)$. But a subset of $N(A)$ is a g-inverse set $\Leftrightarrow$

it is a p-inverse set. Since p is a quotient map, p maps open p-inverse sets to open sets in $L(A)$. The continuity of f follows from the fact that for any subset S of $l^2(A)$, $f^{-1}(S) = p(g^{-1}(S))$. □

Lemma 12. *The map $f^{-1}\colon f(L(A)) \to L(A)$ is continuous.*

Proof. Suppose $g(\boldsymbol{a}^n) \to g(\boldsymbol{a})$ in $l^2(A)$. Then define sets

$$R = g^{-1}g(\boldsymbol{a}) \quad \text{and} \quad R_n = g^{-1}g(\boldsymbol{a}^n) \qquad (n = 1, 2, \ldots).$$

Because g and p have the same fibers and because p is a closed map (open p-inverse sets form a local base at R), it suffices to show that "$R_n \to R$," i.e., for any open set G with $R \subset G$ there is an N such that when $n > N$ then $R_n \subset G$. Suppose this is not the case. Then there is an infinite subset M of N and a sequence $\{\boldsymbol{r}^m | m \in M\}$ of points $\boldsymbol{r}^m \in R_m$ none of which are members of G. There are two possibilities: Case I. The sequence $\{\mathbf{r}^m\}$ has a convergent subsequence, the limit of which must be a point $\boldsymbol{p} \notin R$. But this would contradict the fact that g is continuous. Thus, this case is impossible. Case II. The sequence $\{\boldsymbol{r}^m\}$ has no convergent subsequence. In this case there is some $i \geq 1$ such that the ith components r_i^m, $m \in M$, of the members of $\{\boldsymbol{r}^m\}$ form an infinite set. (Otherwise, we could obtain a subsequence of $\{\boldsymbol{r}^m\}$ that converges.) It follows that $\{\mathbf{r}^m\}$ has a subsequence $\{\mathbf{s}^k\}$ whose ith components form an infinite set containing neither z nor any of the first $i+1$ components of the (at most two) members of R. But then, using the notation (11), for any k and any $b = s_i^k$,

$$1/2^{i+1} \leq |g_b(\boldsymbol{s}^k) - g_b(\boldsymbol{a})| \leq d(g(\boldsymbol{s}^k), g(\boldsymbol{a})).$$

This contradicts $g(\boldsymbol{a}^n) \to g(\boldsymbol{a})$. □

The following theorem is a combination of Lemmas 9, 11, and 12.

Theorem 13. *The map $\alpha \mapsto \{\alpha^b\}$ is a homeomorphism from $L(A)$ into $l^2(A)$.*

7. The web complexes

The imbedding $\alpha \mapsto \{\alpha^b\}$ maps the endpoints of $L(A)$ to the origin and unit basis vectors while expansions of points of $L(A)$ convey position analogous to the binary expansions of points in I.

Fractals appear: Web complexes $M_0 \subset M_1 \subset \cdots$ exist where M_n is the union of all webs in the web complex induced from the n-skeleton of the (possibly infinite-dimensional) simplex Δ^A. One can also think of M_n as those points α in $L(A)$ with some expansion $a_1a_2\cdots$ where the set $\{a_1, a_2, \ldots\}$ has at most $n+1$ members.

The union M of the webs in $\bigcup_n M_n$ is a dense subspace of $L(A)$ since it contains all rational points of $L(A)$. Inside $l^2(A)$ each rational point of $L(A)$ has only finitely many nonzero coordinates, i.e., each rational point is actually in the subspace $\mathscr{E}^A$ of $l^2(A)$. Indeed, on the other hand, since M consists of those points in $L(A)$ of finite character, we see, inside $l^2(A)$, that $M \subset \mathscr{E}^A$. On the other hand, since $M' = L(A) - M$ consists of those points in $L(A)$ with infinite character, we see $M' \subset (l^2(A) - \mathscr{E}^A)$. This subspace M' is also dense in $L(A) = M \cup M'$.

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