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ZERO-DIMENSIONALITY IN COMMUTATIVE RINGS

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ABSTRACT. If $\{R_\alpha\}_{\alpha\in A}$ is a family of zero-dimensional subrings of a commutative ring T, we show that $\bigcap_{\alpha\in A} R_c$ is also zero-dimensional. Thus, if R is a subring of a zero-dimensional subring of T (a condition that is satisfied if and only if a power of rT is idempotent for each $r \in R$), then there exists a unique minimal zero-dimensional subring R^0 of T containing R. We investigate properties of R^0 as an R-algebra, and we show that R^0 is unique, up to R-isomorphism, only if R itself is zero-dimensional.

1. INTRODUCTION

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity element of S belongs to R, and hence is the unity of R. All allusions to the dimension of a ring refer to its Krull dimension; hence a zero-dimensional ring is one in which each proper prime ideal is maximal.

Arapovic in [A$_2$] (see also [H, §3]) considered the problem of imbeddability of a ring in a zero-dimensional ring. The main result of [A$_2$] is Theorem 7. We cite this result below, and in the remainder of this paper, we refer to it, for short, as AIT.

Arapovic's Imbeddability Theorem. *A commutative unitary ring R is imbeddable as a subring of a zero-dimensional ring if and only if there exists a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals of R satisfying the following conditions* (1) *and* (2):

(1) $\bigcap_{\lambda\in\Lambda} Q_\lambda = (0)$.
(2) *For each $a \in R$, there exists a positive integer n_a such that $a^{n_a} \notin \bigcup_\lambda (P_\lambda - Q_\lambda)$, where $P_\lambda = \mathrm{rad}(Q_\lambda)$ is the prime ideal associated with Q_λ.*

Given a family $\{Q_\lambda\}$ satisfying (1) and (2), a first step in the proof of the "if" part of AIT is the imbedding of R in $T = \prod_{\lambda\in\Lambda} (R/Q_\lambda)_{(P_\lambda/Q_\lambda)}$, a product of zero-dimensional quasilocal rings. In this connection we have considered several problems related to AIT, imbeddability, direct products of rings, and the family of Artinian subrings of a commutative ring [GH$_1$, GH$_2$, GH$_3$]. The

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current paper also has close connections with AIT, with its main focus being the families of zero-dimensional subrings and zero-dimensional extension rings of a given ring R. The main results of the paper are

(1) Theorem 2.4, which shows that an arbitrary intersection of zero-dimensional subrings of a given ring is also zero-dimensional;
(2) Theorem 3.1, which (a) provides equivalent conditions for a subring R of a ring T to be contained in a zero-dimensional subring of T and (b) exhibits a set of R-algebra generators for the unique minimal zero-dimensional subring R^0 of T containing R in the case where the conditions of (a) are satisfied;
(3) Theorem 3.3, which describes properties of R^0 as an algebra over R; and
(4) Theorem 4.1, which shows that if R is imbeddable in a zero-dimensional ring, then R^0 is unique up to R-isomorphism only if R itself is zero-dimensional.

In §4, besides the question of uniqueness, we also investigate certain minimal zero-dimensional extension rings of R that arise from canonical imbeddings of R in a zero-dimensional ring via AIT.

If R is a subring of a ring S, we use the term *S-overring of R* to mean a subring of S containing R. If R is a ring, Spec(R) denotes the prime spectrum of R, and if A is a set, $|A|$ denotes the cardinality of A. The symbol $\mathbb{Z}^+$ denotes the set of positive integers.

2. Intersections of zero-dimensional subrings

Let $\{R_\alpha\}_{\alpha\in A}$ be a family of zero-dimensional subrings of a ring T and let $R = \bigcap_{\alpha\in A} R_\alpha$. We show in Theorem 2.4 that R is zero-dimensional. Consequently, R is von Neumann regular if some R_α is von Neumann regular, and R is Artinian if R is reduced and some R_α is Artinian. Lemmas 2.1 and 2.3 are the key results in the proof of Theorem 2.4. A proof of Lemma 2.1 can easily be obtained from [O] or from [Gl], but because of the importance of Lemma 2.1 in the subsequent development and the brevity of its proof, a proof is included here.

Lemma 2.1 ([O, Lemma 2; Gl, Lemma 4.3.9]). *Suppose x and z are elements of the ring R such that $x = x^2z$. If $y = z^2x$, then $x = x^2y$ and $y = y^2x$. Moreover, if $w \in R$ is such that $x = x^2w$ and $w = w^2x$, then $w = y$.*

Proof. To prove the first assertions, note that $x^2y = x^3z^2 = x(x^2z)z = xxz = x^2z = x$ and $y^2x = z^4x^3 = z^3(zx^2)x = z^3x^2 = z^2 \cdot zx^2 = z^2x = y$. For the statement concerning uniqueness, we have $y^2w^2x^3 = y^2\cdot w^2x\cdot x^2 = y^2\cdot w\cdot x^2 = y^2 \cdot x = y$, and a similar argument shows that $y^2w^2x^3 = w$.

Suppose $x \in R$, a ring. If there exists an element $y \in R$ such that $x = x^2y$ and $y = y^2x$ (that is, if xR is idempotent), then following [Gl, p. 137], we call y the *pointwise inverse of* x. We note that in this case x and y generate the same ideal of R, and xy is the unique idempotent generator of that ideal.

Lemma 2.2. *Suppose R is a subring of the ring S. If e is an idempotent element of R, then the ideal eR is contracted from S.*

Proof. It suffices to show that $eS \cap R \subseteq eR$. Thus, suppose $r = es \in eS \cap R$, where $s \in S$. Then $er = es = r \in eR$, as desired.

The hypothesis that R is unitary was not used in the proof of Lemma 2.1 or 2.2, but the unitary hypothesis is used in the proof of the next result.

Lemma 2.3. *Suppose R is a subring of the ring S and $x \in R$ is such that $x^n R = x^{n+1} R$ for some $n \in \mathbb{Z}^+$. Then $x^n S = x^{n+1} S$, and if $m \in \mathbb{Z}^+$ is chosen minimal so that $x^m S = x^{m+1} S$, then $x^m R = x^{m+1} R$.*

Proof. It is clear that $x^n S = x^{n+1} S$; if $m = n$, the statement that $x^m R = x^{m+1} R$ is also clear. If $m < n$, then to prove that $x^m R = x^{m+1} R$, it suffices to show that $x^m \in x^n R$. Let e be the idempotent generator for $x^n R$. Then $x^m S = x^n S = eS$, so $x^m \in eS \cap R$, and Lemma 2.2 shows that $eS \cap R = eR = x^n R$.

Theorem 2.4. *Suppose $\{R_\alpha\}_{\alpha \in A}$ is a family of zero-dimensional subrings of the ring T.*

(a) *$R = \bigcap_{\alpha \in A} R_\alpha$ is zero-dimensional.*
(b) *$|\operatorname{Spec}(R)| \le |\operatorname{Spec}(R_\alpha)|$ for each α.*
(c) *If $x \in R$ and if x is a unit in some R_α, then x is a unit of each R_α, and hence is a unit of R.*

Proof. (a) To show that R is zero-dimensional, it suffices to show that for each $r \in R$, some power of rR is idempotent [W, Proposition 4.2; M, Proposition 2.3; H, Theorem 3.1]. Thus, take $r \in R$ and take $\alpha \in A$. Some power of rR_α is idempotent since R_α is zero-dimensional, so some power of rT is idempotent. Choose $m \in \mathbb{Z}^+$ minimal so that $r^m T$ is idempotent. Then Lemma 2.3 shows that $r^m R_\beta$ is idempotent for each $\beta \in A$. Let $x = r^m$. By Lemma 2.1, x has a pointwise inverse $y_\beta \in R_\beta$. Moreover, since each R_β is a subring of T, Lemma 2.1 also shows that $y_\beta = y_\gamma$ for all β, $\gamma \in A$. Hence if $y = y_\beta$, then $y \in R$ and $x = x^2 y$, so the ideal $xR = r^m R$ is idempotent. This completes the proof of (a).

Statement (b) follows from the fact that minimal primes are contracted from an arbitrary extension ring.

(c) Suppose x is a unit of R_β and a nonunit of R_α. Then x belongs to a maximal ideal M of R_α, and hence x belongs to $M \cap R$, a minimal prime of R. It follows [K, Theorem 84] that x is a zero divisor of R, contrary to the assumption that x is a unit of R_β. We conclude that x is a unit of each R_α and of R, as asserted.

Corollary 2.5. *Suppose R is a subring of a zero-dimensional subring of a ring T. Then there exists a unique minimal zero-dimensional subring R^0 of T containing R.*

Proof. We take R^0 to be the intersection of the family of all zero-dimensional subring of T containing R. Theorem 2.4(a) shows that R^0 is zero-dimensional, and it is clear that R^0 is the unique minimal zero-dimensional subring of T containing R.

We note that in the case where T is zero-dimensional, Corollary 2.5 is the same as Theorem 7 of [A_1]. We will continue to denote the minimal zero-dimensional T-overring of R, when it exists, by R^0, but we observe in §4 that the ring R^0 is not determined up to isomorphism by R alone; R^0 also depends upon T, so notation such as $R^0(T)$ would more accurately reflect the situation that exists. In §4 we consider what we call *canonical minimal zero-dimensional* *extensions* of a ring R that is imbeddable in a zero-dimensional ring. These extensions are determined by a certain type of representation of the zero ideal of R as an intersection of primary ideals. We learned of the next result from Wiegand; this result was also known to Popescu and Vraciu [PV, p. 271].

Corollary 2.6. *If $\{R_\alpha\}_{\alpha\in A}$ is a nonempty family of von Neumann regular subrings of a ring S, then $R=\bigcap_{\alpha\in A} R_\alpha$ is also von Neumann regular.*

Proof. Theorem 2.4 shows that R is zero-dimensional, and clearly R is reduced. Hence R is von Neumann regular [K, Exercise 12, p. 63; G. Exercise 16, p. 111].

Remark 2.7. If R is a ring, Olivier in [O] shows that there exists a Von Neumann regular ring R^* and a homomorphism $f\colon R\to R^*$ such that if $g\colon R\to S$ is any homomorphism with S von Neumann regular, then there exists a unique homomorphism $g^*\colon R\to S$ such that $g=g^*\circ f$. The ring R^* is called the *regular ring associated to* R. Popescu and Vraciu in [PV, §3] observe that R^* can be realized in the form of a minimal zero-dimensional extension ring as follows. Let $\operatorname{Spec}(R)=\{P_i\}_{i\in I}$ and let K_i be the quotient field of R/P_i. Let f_i be the canonical map from R onto R/P_i, and let $f\colon R\to T=\prod_{i\in I} K_i$ denote the product of the maps f_i; we note that f is an imbedding of R if and only if R is reduced. Popescu and Vraciu show that $R^*\cong (f(R))^0$, the minimal zero-dimensional subring of T containing $f(R)$.

The proof of the next result follows easily from Corollary 2.6.

Corollary 2.8. *Suppose $\{R_\alpha\}_{\alpha\in A}$ is a family of zero-dimensional subrings of a reduced ring S and let $R=\bigcap_{\alpha\in A} R_\alpha$. If some R_α is Artinian, then R is also Artinian.*

Proof. The ring R is von Neumann regular by Corollary 2.6. If R_α is Artinian, then R_α contains only finitely many idempotents, [ZS, Chapter IV, §3] and hence R has only finitely many idempotents. Therefore R is a finite direct sum of fields, and R is Artinian.

We remark that, in general, the intersection of two or more Artinian subrings of a ring S need not be Artinian [GH_3, §4].

3. The algebra structure of a minimal zero-dimensional extension

Suppose R is a subring of a ring T. Theorem 3.1 gives equivalent conditions for R to admit a zero-dimensional T-overring, and if these conditions are satisfied, exhibits a set of generators for R^0 as an algebra over R. Theorem 3.3 gives some properties of R^0 as an algebra over R.

Theorem 3.1. *Suppose R is a subring of a ring T.*

(1) *If R admits a zero-dimensional T-overring, then a power of rT is idempotent for each r in R.*

(2) *Conversely, if a power of rT is idempotent for each $r\in R$, then R admits a zero-dimensional T-overring. In fact, if for r in R, $m(r)$ is the smallest positive integer k such that r^kT is idempotent and t_r is the pointwise inverse of $r^{m(r)}$ in T, then $R[\{t_r : r\in R\}]$ is the unique minimal zero-dimensional T-overring of R.*

Proof. (1) If S is a zero-dimensional T-overring of R, then a power of rS, and hence of rT, is idempotent for each r in R.

(2) Let $S = R[\{t_r : r \in R\}]$. To prove that S is zero-dimensional, let P be a proper ideal of S, let $\overline{R} = (R+P)/P \cong R/(P \cap R)$, let $\overline{S} = S/P$, and for $s \in S$, let $\overline{s} = s + P$. We note that $\overline{S} = \overline{R}[\{\overline{t}_r : r \in R\}]$. If $\overline{r}$ is a nonzero element of $\overline{R}$, where $r \in R$, and if $m = m(r)$, then $\overline{r}^m = \overline{r}^{2m}\overline{t}_r$, and since $\overline{S}$ is an integral domain, $\overline{1} = \overline{r}^m \overline{t}_r$. It follows that $\overline{r}$ is a unit of $\overline{S}$, and since $\overline{r}$ is an arbitrary nonzero element of $\overline{R}$, the quotient field K of $\overline{R}$ is a subfield of $\overline{S}$. On the other hand, if $\overline{t}_a$ is a nonzero element of $\overline{S}$, the equation $\overline{t}_a(\overline{t}_a\overline{a}^{m(a)} - \overline{1}) = 0$ implies that $\overline{t}_a = (\overline{a}^{m(a)})^{-1} \in K$, so $\overline{S} = \overline{R}[\{\overline{t}_r\}] \subseteq K$ and equality holds: $\overline{S} = K$. This proves that S/P is a field, so S is zero-dimensional. If U is any zero-dimensional T-overring of R and if $r \in R$, then Lemma 2.3 shows that $r^{m(r)}U$ is idempotent. Hence $r^{m(r)}$ has a pointwise inverse u_r in U, and by uniqueness of pointwise inverses in T, $t_r = u_r \in U$. We conclude that $S = R[\{t_r\}] \subseteq U$, and this completes the proof of Theorem 3.1.

Proposition 3.2. *Suppose R is a subring of a ring T. Assume that $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal nonzero idempotents of R with sum 1. Let $R_i = Re_i$ and $T_i = Te_i$ for each i, so that $R = R_1 \oplus \cdots \oplus R_n$ and $T = T_1 \oplus \cdots \oplus T_n$. Then R admits a zero-dimensional T-overring if and only if R_i admits a zero-dimensional T_i-overring for each i; moreover, if these conditions are satisfied, then the minimal zero-dimensional T-overring R^0 of R is $R_1^0 \oplus \cdots \oplus R_n^0$.*

Proof. If S is a zero-dimensional T-overring of R, then Se_i is a zero-dimensional Te_i-overring of Re_i for each i, and conversely, if S_i is a Te_i-overring of Re_i for each i, then $S_1 \oplus \cdots \oplus S_n$ is a zero-dimensional T-overing of R. This establishes the existence assertion in Proposition 3.2. If R admits a zero-dimensional T-overing, then inclusion $R^0 \subseteq R_1^0 \oplus \cdots \oplus R_n^0$ is clear, and the reverse inclusion holds because $R^0 = R^0e_1 \oplus \cdots \oplus R^0e_n$, where R^0e_i is a zero-dimensional T_i-overring of R_i.

Theorem 3.3. *Suppose R is a subring of a zero-dimensional subring of a ring T and let R^0 be the minimal zero-dimensional T-overing of R.*

(a) *If P is a prime ideal of R^0, then R^0/P is the quotient field of its subring $(R+P)/P \cong R/(P \cap R)$.*

(b) ([A₁, Proposition 8]) *A primary ideal of R^0 is uniquely determined by its contraction to R.*

(c) $|\operatorname{Spec}(R^0)| \leq |\operatorname{Spec}(R)|$; *in particular,* $\operatorname{Spec}(R^0)$ *is finite if* $\operatorname{Spec}(R)$ *is finite. If $H \in \operatorname{Spec}(R)$, then H is contracted from R^0 if and only if H is contracted from T.*

(d) *If* $\operatorname{Spec}(R)$ *is finite, then R^0 is a finitely generated R-algebra.*

Proof. Throughout the proof we use the notation $m(r)$ and t_r, for $r \in R$, as in the statement of (2) of Theorem 3.1; thus $R^0 = R[\{t_r : r \in R\}]$. The assertion in (a) is established in the course of the proof of Theorem 3.1. To prove (b), let Q_1 and Q_2 be primary ideals of R^0 with the same contraction Q to R. Let $P_i = \operatorname{rad}(Q_i)$ and let $P = \operatorname{rad}(Q)$; we have $P = P_i \cap R$. For $r \in R$, we first show that $t_r \in Q_1$ if and only if $t_r \in Q_2$. Thus, assume that $t_r \in Q_1$. Since $r^{m(r)}$ and t_r generate the same ideal of R^0, it follows that $r \in P_1 \cap R = P$, and hence $r \in P_2$. We have $t_r(1 - t_r r^{m(r)}) = 0 \in Q_2$ and $1 - t_r r^{m(r)} \notin P_2$, so

$t_r \in Q_2$. Therefore Q_1 and Q_2 contain exactly the same elements t_r for $r \in R$. Now consider an arbitrary element w of Q_1. We have $w \in R[t_{r_1}, \ldots, t_{r_s}]$ for some finite subset $\{r_i\}_{i=1}^s$ of R. Thus $w = \sum_{i=1}^k a_i t_{r_1}^{e_{i1}} \cdots t_{r_s}^{e_{is}}$, where each $a_i \in R$. We wish to show that $w \in Q_2$, and because Q_1 and Q_2 contain exactly the same elements t_r, we may assume without loss of generality that each t_{r_i} belongs to neither Q_1 nor Q_2. To simplify the notation, let $b_i = r_i^{m(r_i)}$ and $u_i = t_{r_i}$ for each i. Since b_i and u_i generate the same ideal of R^0, no b_i is in Q_1 or Q_2. The equation $b_i(1 - u_i b_i) = 0$ then implies that $1 - u_i b_i \in P_1 \cap P_2$, and since $u_i b_i$ and $1 - u_i b_i$ are idempotent, it follows that $1 - u_i b_i \in Q_1 \cap Q_2$. Let $e_1 = \max\{e_{1i}, e_{2i}, \ldots, e_{ki}\}$ for $1 \le i \le s$ and let $b = b_1^{e_1} \cdots b_s^{e_s}$. Since $b_i u_i \equiv 1(\bmod Q_1 \cap Q_2)$ for each i, we have $bw = \sum_{i=1}^k a_i b_1^{e_1} u_1^{e_{i1}} \cdots b_s^{e_s} u_s^{e_{is}} \equiv c = \sum_i^k a_i b_1^{e_1 - e_{i1}} \cdots b_s^{e_s - e_{is}} \pmod{Q_1 \cap Q_2}$. Now $w \in Q_1$, so $bw \equiv 0 \pmod{Q_1}$, and hence $c \in Q_1 \cap R = Q$. Consequently, $c \in Q_2$, $bw \in Q_2$, and thus $w \in Q_2$ since $b \notin P_2$. We have proved that $Q_1 \subseteq Q_2$, and by symmetry, it follows that $Q_1 = Q_2$.

The first assertion in (c) follows immediately from (b). For the second, it is clear that H is contracted from R^0 if it is contracted from T, and the converse holds because each prime of R^0 is contracted from T.

(d) If $\operatorname{Spec}(R)$ is finite, then so is $\operatorname{Spec}(R^0)$, and hence the set of idempotents of R^0 is finite. Express R^0 as a direct sum $R^0 e_1 \oplus \cdots \oplus R^0 e_n$, where each e_i is a nonzero idempotent, each $R^0 e_i$ is indecomposable as a ring, and hence each $R^0 e_i$ is quasilocal. Replacing R by $R[e_1, \ldots, e_n] = Re_1 \oplus \cdots \oplus Re_n$, we see that in view of Proposition 3.2, it suffices to prove (d) under the assumption that R^0 is quasilocal. In this case let M be the maximal ideal of R^0 and let $P = M \cap R$. Then P is the unique minimal prime of R. If $x \in R$ is chosen in each prime ideal of R except P, then x is a unit of R^0 and $R[x^{-1}]$ is a zero-dimensional subring of R^0. By definition of R^0, it follows that $R^0 = R[x^{-1}]$, so R^0 is finitely generated over R, as we wished to show.

Remark 3.4. In connection with part (b) of Theorem 3.3, we remark that if Q is a primary ideal of R^0, then Q need not be the extension to R^0 of $Q \cap R$. in fact, if P is a prime ideal of R^0, P need not be contained in $\operatorname{rad}((P \cap R)R^0)$. For example, let (R, M) be a one-dimensional local domain and let $T = K \oplus (R/M)$, where K is the quotient field of R and where R is imbedded in T by the inclusion map on the first component and the natural homomorphism of R onto R/M in the second component. If $m \in M$, $m \neq 0$, then the pointwise inverse of m in T is $(m^{-1}, \overline{0})$ and this element is in R^0. If x is any element of K, then x is of the form y/n, where $y \in R$ and $n \in M$. Hence $(x, \overline{0}) = (y, \overline{y})(n^{-1}, \overline{0}) \in R^0$, and from this it follows easily that $R^0 = T$. Let P be the prime ideal $((0), R/M)$ of T. Then $P \cap R = (0)$, and $P \not\subseteq \operatorname{rad}((0)R^0) = (0)$.

Remark 3.5. Let the notation and hypothesis be as in the statement of Theorem 3.3. In the case where T is zero-dimensional, Arapovic proves that R^0 is the total quotient ring of an integral extension of R [A_1, Theorem 6]. The proof of Theorem 3.3(d) implicitly proves this result in the case where $\operatorname{Spec}(R)$ is finite; we can see that the result holds in general (without the assumption that T is zero-dimensional) as follows: for $r \in R$, the element $f_r = r^{m(r)} t_r$ of R^0 is idempotent, so $S = R[\{f_r : r \in R\}]$ is an integral extension of R. The element

$s_r = r^{m(r)} + (1 - f_r)$ of S belongs to no maximal ideal of R^0 because $r^{m(r)}$ and f_r generate the same ideal of R^0, and hence s_r is a unit of R^0. It follows that $S[\{s_r^{-1} : r \in R\}] \subseteq R^0 = R[\{t_r : r \in R\}]$, but since $t_r = f_r/s_r$, the reverse inclusion also holds. Thus $R^0 = S[\{s_r^{-1} : r \in R\}]$ is the total quotient ring of S. (Since $S[\{s_r^{-1}\}]$ is zero-dimensional, it is equal to its own total quotient ring.)

4. Uniqueness of minimal zero-dimensional extensions and some canonical minimal extensions related to Arapovic's Imbeddability Theorem

If the ring R is imbeddable in a zero-dimensional ring, then R admits a minimal zero-dimensional extension R^0. Theorem 4.1 shows, however, that R^0 is uniquely determined up to R-isomorphism only if R is zero-dimensional, and hence $R = R^0$. Proposition 4.4 deals with calculation of the ring R^0 in case the zero ideal of R is a finite intersection of primary ideals. After proving Proposition 4.4, we turn our attention to some canonical minimal extensions of R that arise from AIT. (Recall that AIT designates the theorem of Arapovic cited in the introduction.) These extensions are determined by a family of primary ideals of R satisfying conditions (1) and (2) of AIT; in general there may be many such families in R, and this is another factor that lends credence to the frequent lack of uniqueness of R^0.

Theorem 4.1. *Suppose the ring R is imbeddable in a zero-dimensional ring S and that R^0 is unique up to R-isomorphism.*

(1) *If Q is a P-primary ideal of R, then there exists a prime ideal P^* of R^0 and a P^*-primary ideal Q^* of R^0 such that P^* and Q^* lie over P and Q, respectively.*

(2) *If P is a prime ideal of R, then there exists a unique minimal P-primary ideal of R.*

(3) *R is zero-dimensional.*

Proof. (1) Let T be the zero-dimensional ring $S \oplus R_P/QR_P$. The ideal $I = S \oplus (0)$ of T is primary for the ideal $J = S \oplus PR_P/QR_P$. Moreover, if R is considered as a subring of T via the diagonal imbedding, then P and Q are the contractions to R of J and I, respectively. Let U be the minimal zero-dimensional T-overring of R. Then $I \cap U$ is $J \cap U$-primary in U, and these ideals lie over Q and P in R. Because of uniqueness of R^0 up to R-isomorphism, the conclusion of (1) then follows.

(2) Let P^* be a prime ideal of R^0 lying over P in R. Since R^0 is zero-dimensional, the P^*-primary component Q^* of (0) in R^0 is the unique minimal P^*-primary ideal of R^0. Let $Q = R \cap Q^*$. We claim that each P-primary ideal Q_0 of R contains Q. Thus, by (1), there exists a P^*-primary ideal Q_0^* of R^0 lying over Q_0 in R. Since Q_0^* contains Q^*, it follows that Q_0 contains Q, as claimed.

(3) Suppose that $\dim(R) > 0$, and choose prime ideals P_0, P_2 of R with $P_0 < P_2$. Choose $x \in P_2 - P_0$, let P be a minimal prime ideal of $P_0 + (x)$, and let P_1 be a prime of R that contains P_0, is contained in P, and is maximal with respect to failure to contain x. Then $P_1 < P$, and there is no prime ideal of R properly between these two. Thus [G, (17.4)], the intersection of the set of

P-primary ideals of R that contain P_1 is P_1. Therefore if Q is the minimal P-primary ideal of R, then Q is contained in P_1, and this contradicts the fact that $\operatorname{rad}(Q) = P$. This completes the proof.

Remark 4.2. If a ring R is a subring of a zero-dimensional quasilocal ring (T, N), then $N \cap R = P$ is the unique minimal prime of R. Moreover, if $x \in R - P$, then x is a unit of T and hence is a regular element of R. Therefore the total quotient ring of R is the localization R_P, and R_P is the minimal zero-dimensional subring of T containing R.

We note that Remark 4.2 is the special case of the next result where $n = 1$.

Proposition 4.3. *Suppose n is a positive integer, T is a zero-dimensional ring with exactly n minimal prime ideals, and R is a subring of T with exactly n minimal primes. Then the total quotient ring of R is the minimal zero-dimensional extension of R in T.*

Proof. Let $\{P_i\}_{i=1}^n$ be the set of minimal primes of T. Since each minimal prime of R is contracted from T, it follows that $\{P_i \cap R\}_{i=1}^n$ is the set of (distinct) minimal primes of R. Let S be any zero-dimensional T-overring of R. Then $\{P_i \cap S\}_{i=1}^n$ is the set of prime ideals of S by the argument just given, and these ideals are distinct since their contractions to R are distinct. If $r \in R - (\bigcup_{i=1}^n (P_i \cap R))$, then $r \in S - (\bigcup_{i=1}^n (P_i \cap S))$, and hence r is a unit of S. Therefore r is a regular element of R and $\bigcup_{i=1}^n (P_i \cap R)$ is the set of zero divisors of R. Thus the total quotient ring of R is zero-dimensional and is contained in S. This completes the proof.

Proposition 4.4. *Let R be a ring in which* (0) *has a finite primary decomposition. Let* $(0) = Q_1 \cap \cdots \cap Q_n$ *be a shortest primary decomposition of* (0), *where Q_i is a P_i-primary ideal in R. Consider the ring $T = (R_{P_1}/Q_1R_{P_1}) \oplus \cdots \oplus (R_{P_n}/Q_nR_{P_n})$ and let $\phi\colon R \to T$ be the canonical imbedding of R in T. Then any proper subring of T containing $\phi(R)$ is of positive dimension, and hence $\phi(R)^0 = T$.*

Proof. We may identify R with $\phi(R)$ and regard R as a subring of T. Let S be a zero-dimensional T-overring of R. Because the zero ideal of T admits a finite primary decomposition, the same is tue for the zero ideal of S, and any such primary decomposition of (0) in S contracts to R to give a primary decomposition of (0) in R. Therefore S must have at least n prime ideals. It follows that the map from $\operatorname{Spec}(T)$ to $\operatorname{Spec}(S)$ induced by the inclusion map $S \subseteq T$ is a bijection, and S has exactly n prime ideals. Let W_i denote the prime ideal of $T = (R_{P_1}/Q_1R_{P_1}) \oplus \cdots \oplus (R_{P_n}/Q_nR_{P_n})$ consisting of all tuples $(t_1, \ldots, t_n)$ for which $t_i = 0$. Note that $T_{W_i} \cong R_{P_i}/Q_iR_{P_i}$. Let ϕ_i denote the canonical map of R into $R_{P_i}/Q_iR_{P_i} \cong T_{W_i}$. By permutability of localization and residue class ring formation, we have $R_{P_i}/Q_iR_{P_i} \cong (R/Q_i)_{(P_i/Q_i)}$ and $\phi_i(R) = R/Q_i \subseteq T_{W_i}$. Moreover, if $W_i \cap S = U_i$, then $R/Q_i \subseteq S_{U_i} \subseteq T_{W_i}$. Since T_{W_i} is the total quotient ring of R/Q_i and since S_{U_i} is zero-dimensional, we have $S_{U_i} = T_{W_i}$. Because this is true for each prime U_i of S, it follows that $S = T$.

The proof of Theorem 4.1 uses the fact that the isomorphisms of minimal zero-dimensional extensions under consideration are R-algebra isomorphisms.

With additional hypothesis on R, we show in Proposition 4.5 that R admits two minimal zero-dimensional extensions that are not isomorphic as rings; in particular, the hypothesis of Proposition 4.5 is satisfied if R is Noetherian.

Proposition 4.5. *Suppose R is a ring of positive dimension and assume that (0) in R is a finite intersection of primary ideals. Let $(0) = Q_1 \cap \cdots \cap Q_n$ be a shortest primary decomposition of (0) in R, where $\mathrm{rad}(Q_i) = P_i$, $i = 1, \ldots, n$. Assume that either*

(1) *there exists $P \in \mathrm{Spec}(R) - \{P_i\}_{i=1}^n$, or*
(2) *each of the primary ideals Q_i contains a power of its radical P_i.*

Then there exist two minimal zero-dimensional extensions of R that are not isomorphic as rings.

Proof. If $T_1 = R_{P_1}/Q_1R_{P_1} \oplus \cdots \oplus R_{P_n}/Q_nR_{P_n}$, then Proposition 4.4 shows that T_1 is minimal zero-dimensional extension of R; moreover, T_1 has exactly n prime ideals, and these primes lie over $P_1, \ldots, P_n$ in R. Suppose there exists $P \in \mathrm{Spec}(R) - \{P_i\}_{i=1}^n$, let K be the quotient field of R/P, and let T be the zero-dimensional ring $T_1 \oplus K$. If R is considered as a subring of T under the diagonal imbedding, then each of the ideals $P, P_1, \ldots, P_n$ is contracted from T, and hence, by Theorem 3.3(c), from the minimal zero-dimensional extension R^0 of R in T. Consequently, $|\mathrm{Spec}(R^0)| > n$ and R^0 is not isomorphic to T_1.

Suppose $\mathrm{Spec}(R) = \{P_i\}_{i=1}^n$. If $P_i^{k_i} \subseteq Q_i$, and if k is the maximum of the integers $k_1, \ldots, k_n$, it follows that T_1 has the property that each primary ideal of T_1 contains the kth power of its radical; we exhibit a minimal zero-dimensional extension of R that does not share this property of T_1. Thus, assume that the labeling is such that P_1 is maximal in R and $P_1 > P_2$. Choose $x \in P_1 - \bigcup_{i=2}^n P_i$. Then xR is P_1-primary. The powers of xR properly descend, for an equality $(x^t)R = (x^{t+1})R$ would lead to an equation $x^t(1-rt) = 0$, where neither x^t nor $1 - rx$ is in P_2. If $t > k$, then $x^t \in Q_1$, and it follows that $(0) = x^tR \cap Q_2 \cap \cdots \cap Q_n$ is a shortest primary decomposition of (0). If T_2 is the minimal zero-dimensional extension of R arising from this decomposition as in Proposition 4.4, then x^tT_2 is a primary ideal of T_2 that does not contain the kth power of its radical.

We next turn our attention to what we call *canonical minimal zero-dimensional extensions* of a ring that arise from AIT. Thus, suppose R is imbeddable in a zero-dimensional ring. Then there exists a family $\{Q_\lambda\}$ of primary ideals of R satisfying conditions (1) and (2) of AIT. Now R is canonically imbedded in $T = \prod_\lambda (R/Q_\lambda)_{(P_\lambda/Q_\lambda)}$, and condition (2) of AIT implies that a power of rT is idempotent for each r in R. Thus T contains a minimal zero-dimensional extension of R, and we refer to a minimal zero-dimensional extension of R that arises in this way as a *canonical extension.* If S is any minimal zero-dimensional extension of R, then (0) in S is an intersection of a family of primary ideals of S. The contraction to R of this family of primary ideals of S is a family $\{Q_\lambda\}$ of primary ideals of R satisfying conditions (1) and (2) of AIT. We remark that the canonical minimal zero-dimensional extension of R associated to the family $\{Q_\lambda\}$ is R-isomorphic to S. In general, canonical extensions of R are not unique, but Remark 4.6 does allow us to make one significant simplifying assumption in regard to the family $\{Q_\lambda\}$.

Remark 4.6. If $\{Q_\alpha\}_{\alpha\in A}$ is a family of P-primary ideals of a ring R and if $Q = \bigcap_{\alpha\in A} Q_\alpha$, it is straightforward to show that Q is P-primary if and only if $P \subseteq \mathrm{rad}(Q)$. We observe that if $\{Q_\alpha\}_{\alpha\in A}$ is a subfamily of a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals satisfying condition (2) of AIT, the Q_α all having the same associated prime P, then the inclusion $P \subseteq \mathrm{rad}(Q)$ holds. To see this, take $a \in P$ and choose $n_a \in \mathbb{Z}^+$ so that $a^{n_a} \notin \bigcup_\lambda (P_\lambda - Q_\lambda)$; then $a^{n_a} \in Q = \bigcap_{\alpha\in A} Q_\alpha$, so $a \in \mathrm{rad}(Q)$ as desired. This observation means that in considering families $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals satisfying conditions (1) and (2) of AIT, there is no loss of generality in assuming that $\mathrm{rad}(Q_\lambda) \neq \mathrm{rad}(Q_\mu)$ for λ, $\mu \in \Lambda$ and $\lambda \neq \mu$.

Proposition 4.7. *Assume that R is a ring imbeddable in a zero-dimensional ring, and let $\mathscr{S} = \{Q_\lambda\}_{\lambda\in\Lambda}$ be a family of primary ideals of R satisfying conditions (1) and (2) of AIT. Let $P_\lambda = \mathrm{rad}(Q_\lambda)$ and consider R as a subring of $T = \prod_\lambda (R_\lambda/Q_\lambda)_{(P_\lambda/Q_\lambda)}$ via the canonical imbedding. If $\{P_\lambda\}_{\lambda\in\Lambda}$ is infinite, then R^0 contains infinitely many idempotents.*

Proof. By definition, if $a \in R$, then the idempotent generator e_a of the ideal aT is the tuple whose λth coordinate is 1 or 0, according as a is not, or is, in Q_λ. Moreover, $e_a \in R^0$ by Theorem 3.1(2). Let $n \in \mathbb{Z}^+$, let $\{P_1, P_2, \dots, P_n\}$ be a set of n distinct elements of $\{P_\lambda\}$, and assume that the ordering is such that P_i is minimal in $\{P_i, P_{i+1}, \dots, P_n\}$ for $1 \leq i \leq n$. This means that if we choose $a_1 \in Q_1 \cap \cdots \cap Q_n$, $a_2 \in (Q_2 \cap \cdots \cap Q_n)\backslash Q_1, \dots, a_n \in Q_n\backslash Q_{n-1}$, then the idempotents $e_{a_1}, e_{a_2}, \dots, e_{a_n}$ in R^0 are distinct. Since n is arbitrary, it follows that R^0 has infinitely many idempotents.

Remark 4.8. Suppose R is a ring imbeddable in a zero-dimensional ring. If $\mathrm{Spec}(R)$ is infinite, then Proposition 4.7 can be used as follows to show that R has a minimal zero-dimensional extension R^0 with infinitely many idempotents: There exists a family $\{Q_\lambda\}$ of primary ideals of R satisfying (1) and (2) of AIT, and the family $\mathscr{T} = \{Q_\lambda\} \cup \mathrm{Spec}(R)$ satisfies the same conditions. Using $\mathscr{T}$ as the family $\mathscr{S}$ in the statement of Proposition 4.7, we obtain an R^0 with infinitely many idempotents. On the other hand, if $\mathrm{Spec}(R)$ is finite, then Remark 4.6 shows that in any family $\{Q_\lambda\}$ of primary ideals satisfying the conditions of AIT, we may assume without loss of generality that distinct ideals Q_λ have distinct radicals, and hence that $\{Q_\lambda\}$ is finite. In this case the ring $T = \prod_\lambda (R/Q_\lambda)_{(P_\lambda/Q_\lambda)}$ of Proposition 4.7 has only finitely many idempotents, and hence R^0 has only finitely many idempotents.

Theorem 4.9. *Let R be a ring with finite spectrum of cardinality m. The following conditions are equivalent.*

(1) *R is imbeddable in a zero-dimensional ring.*
(2) *The zero ideal of R is a finite intersection of primary ideals.*
(3) *R admits a minimal zero-dimensional extension ring with at most m prime ideals.*

Proof. That (1) implies (2) follows from AIT and Remark 4.6. Proposition 4.4 shows that (2) implies (3), and the implication (3) $\Rightarrow$ (1) is obvious.

In relation to Theorem 4.9 it would be interesting to know whether a ring R which is a subring of a zero-dimensional ring and which has Noetherian spectrum necessarily has the property that (0) in R is a finite intersection of primary ideals of R.

If $\{Q_\lambda\}$ is a family of primary ideals of a ring R satisfying the conditions of AIT, then the ring $T = \prod_\lambda (R/Q_\lambda)_{(P_\lambda/Q_\lambda)}$ need not itself be zero-dimensional. For example, we show in Result 4.11 that there exist zero-dimensional rings not imbeddable in any zero-dimensional product of quasilocal rings. Theorem 4.10 gives equivalent conditions for a ring to be imbeddable in a zero-dimensional product of quasilocal rings. The statement of Theorem 4.10 uses the notation $\eta(\cdot)$, defined as follows (cf. [GH$_1$, p. 633]). Let R be a ring with nilradical $N(R)$. If $x \in N(R)$, then $\eta(x)$ is the index of nilpotency of x—that is, $\eta(x) = k$ if $x^k = 0$ but $x^{k-1} \neq 0$. We define $\eta(R)$ to be $\sup\{\eta(x) : x \in N(R)\}$; if the set $\{\eta(x) : x \in N(R)\}$ is unbounded, then we write $\eta(R) = \infty$. For an ideal I of R, $\eta(I)$ is defined to be $\eta(R/I)$; the definition amounts to the following: if $J = \operatorname{rad}(I)$, then $\eta(I) = \inf\{n \in \mathbb{Z}^+ : x^n \in I \text{ for each } x \in J\}$, where it is understood that $\eta(I) = \infty$ if there is no positive integer n such that $x^n \in I$ for each $x \in J$.

Theorem 4.10. *For a ring R, the following conditions are equivalent.*

(1) *R is imbeddable in a zero-dimensional product of quasilocal rings.*

(2) *The zero ideal of R is representable as an intersection of a family $\{Q_i\}_{i\in I}$ of primary ideals such that, for some $k \in \mathbb{Z}^+$, $\{i \in I : \eta(Q_i) > k\}$ is finite.*

Proof. (1) $\Rightarrow$ (2). Suppose R is imbeddable in S, where S is the product of a family $\{(R_i, M_i)\}_{i\in I}$ of quasilocal rings and S is zero-dimensional. By [GH$_2$, Theorem 3.4], there exists $k \in \mathbb{Z}^+$ such that $\{i \in I : \eta(R_i) > k\}$ is finite. Let U_i be the primary ideal of S consisting of all tuples with ith coordinate 0; $\operatorname{rad}(U_i) = W_i$ consists of all tuples whose ith coordinate is in M_i. Clearly $\bigcap_{i\in I} U_i = (0)$ and $\eta(U_i) = \eta(R_i)$. Let Q_i and P_i denote the contractions to R of U_i and W_i, respectively. Then Q_i is P_i-primary, $\bigcap_{i\in I} Q_i = (0)$, and since it is clear that $\eta(Q_i) \leq \eta(U_i) = \eta(R_i)$, the required condition in (2) is satisfied by the family $\{Q_i\}_{i\in I}$.

(2) $\Rightarrow$ (1). Suppose the ideal (0) of R is an intersection of a family $\{Q_i\}_{i\in I}$ of primary ideals satisfying the condition in (2). Then R is imbedded in $S = \prod_{i\in I}(R/Q_i)_{(P_i/Q_i)}$, each $(R/Q_i)_{(P_i/Q_i)}$ is zero-dimensional and quasilocal, and $\eta((R/Q_i)_{(P_i/Q_i)}) = \eta(Q_i)$ for each i. Again using Theorem 3.4 of [GH$_2$], we conclude that S is zero-dimensional.

Result 4.11. *Let $\{X_i\}_{i=1}^\infty$ be a set of indeterminates over a field F and let I be the ideal of $F[\{X_i\}_{i=1}^\infty]$ generated by the set $\{X_i(X_i+1)^i\}_{i=1}^\infty$. Then $R = F[\{X_i\}_{i=1}^\infty]/I$ is zero-dimensional, but no family of primary ideals of R satisfies condition* (2) *of Theorem* 4.9, *and hence R is not imbeddable in a zero-dimensional product of quasilocal rings.*

Proof. If A is any subset of $\mathbb{Z}^+$, let A' denote the complement of A in $\mathbb{Z}^+$. We note that $M_A = (\{X_i : i \in A\} \cup \{X_i + 1 : i \in A'\})$ is a maximal ideal of $F[\{X_i\}_{i=1}^\infty]$ containing I. Any prime ideal P of $F[\{X_i\}_{i=1}^\infty]$ containing I is equal to some M_A, for since $X_i(X_i+1)^i \in I \subseteq P$, then P contains one and only one of X_i or $X_i + 1$ for each $i \in \mathbb{Z}^+$. Hence if $B = \{i \in \mathbb{Z}^+ : X_i \in P\}$, then $M_B \subseteq P$, so $M_B = P$. Therefore R is zero-dimensional.

Suppose $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a family of primary ideals of $F[\{X_i\}_{i=1}^\infty]$ such that $I = \bigcap_\lambda Q_\lambda$. Let $P_\lambda = \operatorname{rad}(Q_\lambda)$ and fix $i \in \mathbb{Z}^+$. We partition Λ into subsets Λ_1

and Λ_2, where $\Lambda_1 = \{\lambda \in \Lambda : X_i \in P_\lambda\}$ and $\Lambda_2 = \{\lambda \in \Lambda : X_i + 1 \in P_\lambda\}$. We claim that there exists $\lambda \in \Lambda_2$ such that $(X_i+1)^{i-1} \notin Q_\lambda$. Suppose not. Then $(X_i+1)^{i-1} \in \bigcap_{\lambda\in\Lambda_2} Q_\lambda$. Moreover, if $\lambda \in \Lambda_1$, then $X_i(X_i+1)^i \in Q_\lambda$ and $(X_i+1)^i \notin P_\lambda$, so $X_i \in Q_\lambda$. Therefore $X_i(X_i+1)^{i-1} \in (\bigcap_{\lambda\in\Lambda_1} Q_\lambda)\cap(\bigcap_{\lambda\in\Lambda_2} Q_\lambda) = I$. This leads to a contradiction, as follows. If $X_i(X_i+1)^{i-1}$ were in I, then under the $F[X_i]$-homomorphism of $F[\{X_i\}_{i=1}^\infty]$ to $F[X_i]$ that maps X_j to 0 if $j \neq i$, we would have $X_i(X_i+1)^{i-1} \in X_i(X_i+1)^i F[X_i]$, which is not the case. Therefore $(X_i+1)^{i-1} \notin Q_\lambda$ for some $\lambda \in \Lambda_2$, so $\eta(Q_\lambda) \geq i$ for some $\lambda \in \Lambda_2$. Since i is arbitrary, we conclude that $\{Q_\lambda\}$ does not satisfy condition (2) of Theorem 4.10, as we wished to show.

If R is as in the statement of Result 4.11, then the proof of that result shows that the prime ideals of R are in one-to-one correspondence with the subsets of $\mathbb{Z}^+$, and hence $\mathrm{Spec}(R)$ is uncountable. We can obtain an example of a ring with the same properties as described in (4.11) and with countable spectrum as follows. Let F be a field, let t be an indeterminate over F, and let S be the subring of $T = \prod_{n=1}^\infty F[t]/(t^n)$ generated by F and the direct sum ideal $J = \bigoplus_{n=1}^\infty F[t]/(t^n)$ of T. Since S is integral over F, it is zero-dimensional. Moreover, $\mathrm{Spec}(S) = \{J\} \cup \{P_i\}_{i=1}^\infty$, where P_i consists of all tuples with ith coordinate in $(t)/(t^i)$. If Q_i is the P_i-primary ideal consisting of all tuples with 0 in the ith coordinate (hence $Q_i = P_i^i$), then as in Example 4 of [GH$_1$], each Q_i belongs to each family $\{Q_\lambda\}$ of primary ideals of S with intersection (0). Since $\eta(Q_i) = i$, it follows that S too has the property that it is not imbeddable in a zero-dimensional product of quasilocal rings.

Remark 4.12. Even though AIT gives necessary and sufficient conditions in order that a ring R be a subring of a zero-dimensional ring, it is sometimes difficult in practice to apply. For example, if p is a fixed prime integer, $T = \prod_{n=1}^\infty \mathbb{Z}/p^n\mathbb{Z}$ and $I = \bigoplus_{n=1}^\infty \mathbb{Z}/p^n\mathbb{Z}$ is the direct sum ideal in T, then Roger Wiegand has asked whether the factor ring $R = T/I$ is a subring of a zero-dimensional ring. It is known that T is not a subring of a zero-dimensional ring [GH$_1$, Theorem 3], but the proof involves the irredundance of certain primary ideals in any representation of (0) in T as an intersection of primary ideals, and does not apply to a ring such as R.

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Added in proof

Ken Goodearl has called to our attention the paper by W. D. Burgess and P. Menal, *On strongly π-regular rings and homomorphisms into them*, Comm. Algebra **16** (1988), 1701–1725. Theorem 1.3 of this paper implies that an intersection of zero-dimensional commutative rings is again zero-dimensional. Goodearl has also informed us that the result is in the (unpublished) 1977 dissertation of Dischinger.

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