HANDLEBODY COMPLEMENTS IN THE 3-SPHERE:
A REMARK ON A THEOREM OF FOX

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Abstract. Let $W$ be a compact 3-dimensional submanifold of $S^3$, and $C$ be a collection of disjoint simple closed curves on $\partial W$. We give necessary and sufficient conditions (one extrinsic, one intrinsic) for $W$ to have an embedding in $S^3$ so that $S^3 - W$ is a union of handlebodies, and $C$ contains a complete collection of meridia for these handlebodies.

A theorem of Fox [F] says that any connected compact 3-dimensional submanifold $W$ of $S^3$ is homeomorphic to the complement of a union of handlebodies in $S^3$. Call a collection $C$ of simple closed curves in $\partial W$ proto-meridinal if there is some embedding of $W$ in $S^3$ so that the complement of $W$ is a union of handlebodies and $C$ contains a complete collection of meridia for this handlebody. It is natural to ask whether a given collection $C$ of simple closed curves in $\partial W$ is proto-meridinal. This is a difficult question to answer. For example, if $\partial W$ is a torus, either $W$ is a solid torus or only one curve in $\partial W$ is proto-meridinal. The proof is essentially the celebrated solution by Gordon and Luecke [GL] of the knot complement conjecture.

For genus($\partial W$) > 1 the problem seems intractable. But it does make sense to ask the following question: What conditions on the given embedding of $W$ in $S^3$ (called extrinsic conditions) and what conditions on the pair $(W, C)$ itself (called intrinsic conditions) suffice to guarantee that $C$ is proto-meridinal? The general goal is to discover intrinsic conditions that allow extrinsic conditions to be weakened. Here it is shown that a simple intrinsic condition allows one to reduce the extrinsic requirements to a simple condition on $C$ regarded as a link in $S^3$.

Let $C$ be a collection of disjoint simple closed curves in $\partial W \subset W \subset S^3$. The normal direction to $\partial W$ in $S^3$ determines (up to orientation) a framing of the normal bundle $\eta(C)$ of $C$ in $S^3$. Suppose $C$ is the unlink in $S^3$ and $\Delta$ is a collection of disjoint disks that $C$ bounds. Then the normal of $C$ into $\Delta$ also determines (up to orientation) a framing on $\eta(C)$. If the framings coincide (up to orientation) then we say that $C$ is a framed unlink.

A collection of disjoint simple closed curves in the boundary of a handlebody
is a complete collection of meridia if it bounds a set of disjoint disks whose complementary closure is the 3-ball.

**Theorem.** Suppose \( W \) is a connected compact 3-dimensional submanifold of \( S^3 \) and \( C \) is a family of disjoint simple closed curves in \( \partial W \) that is a framed unlink in \( S^3 \). Then \( C \) is proto-meridinal if and only if \( H_2(W, \partial W - C) \to H_2(W, \partial W) \) is trivial.

**Proof.** The condition on \( H_2 \) can be stated geometrically: Any properly embedded surface in \( W \) whose boundary is disjoint from \( C \) must be separating.

One direction is easy. Suppose \( S^3 - W \) is a union \( H \) of handlebodies and \( C \) contains a complete set of meridia for \( H \). Then any simple closed curve in \( \partial W - C \) bounds a disk in \( H \). Hence any properly embedded surface \( S \) in \( W \) whose boundary lies in \( \partial W - C \) can be capped off to give a closed surface in \( S^3 \), which must be separating. Then \( S \) is separating in \( W \).

For the other direction, we first find a compressing disk \( E \) for \( \partial W \) in \( S^3 - W \) such that \( \partial E \) is disjoint from \( C \). Recall that \( C \) is the framed unlink. Let \( \Delta \) denote a disjoint collection of disks bounded by \( C \) in \( S^3 \), chosen to minimize the number of components of intersection with \( \partial W \). Since the framing of \( \eta(C) \), given by the normal into \( \Delta \) coincides with the framing given by the normal to \( \partial W \) in \( S^3 \), \( \Delta \) can be put in general position with respect to \( \partial W \) so that \( \Delta \cap \partial W \) consists of a set of simple closed curves in \( \Delta \), possibly just \( C \). Choose \( \Delta \) so as to minimize the number of components of \( \Delta \cap \partial W \). Then an innermost circle of intersection of \( \partial W \) with \( \Delta \) (or a component of \( \partial \Delta \)) is an essential circle in \( \partial W \) that is disjoint from \( \partial \Delta \) = \( C \) and bounds a disk \( E \) in \( S^3 - \partial W \).

We now proceed by induction on \( \beta_1 - \beta_0 \), where \( \beta_i \) is the \( i \)th betti number of the union of all nonspherical components of \( \partial W \). If \( \beta_1 - \beta_0 = 0 \), then \( \partial W \) consists of spheres and there is nothing to prove. The hypothesis is that \( H_2(W, \partial W - C) \to H_2(W, \partial W) \) is trivial. With no loss we may assume all curves of \( C \) are essential in \( \partial W \). Consider the following possibilities for \( E \):

**Case 1.** \( E \) lies in \( W \). Since \( H_2(W, \partial W - C) \to H_2(W, \partial W) \) is trivial, \( E \) is separating, and so decomposes \( W \) into the boundary connected sum of two manifolds \( W_1 \) and \( W_2 \), each with boundary of lower genus than that of \( W \). By inductive assumption, each \( W_i \) can be embedded in the 3-sphere so that \( S^3 - W_i \) is a union of handlebodies \( H_i \) in which some subset \( C_i \) of \( C \cap \partial W_i \) bounds a complete collection of meridia. Then the connected sum of these spheres along 3-balls bisected by the equatorial disk \( E \subset \partial W_i \) gives an embedding of \( W \) in \( S^3 \) whose complement is the boundary sum of \( H_1 \) and \( H_2 \), hence a union of handlebodies, in which \( C_1 \cup C_2 \subset C \), is a complete collection of meridia.

**Case 2.** \( E \) lies in \( S^3 - W \). Let \( W' \) be the manifold obtained from \( W \) by attaching a 2-handle \( \eta(E) \) along \( E \). Let \( C' \) be the subcollection of \( C \) that remains essential in \( \partial W' \). Suppose there were a nonseparating surface \( S' \) in \( W' \) with \( \partial S' \subset \partial W' - C' \). Any element of \( C - C' \) is inessential in \( \partial W' \) so we may isotope \( \partial S' \) to be disjoint from \( C \). By general position we may take \( S' \cap \eta(E) \) to be disks parallel to \( E \). Then \( S = S' \cap W \) would be a nonseparating surface in \( W \) with \( \partial S \subset \partial W - C \). We conclude that such a surface \( S' \) cannot exist; i.e., \( H_2(W', \partial W' - C') \to H_2(W', \partial W') \) is trivial. By induction, there is an embedding of \( W' \) in \( S^3 \) so that \( S^3 - W' \) is a union of handlebodies \( H' \) in which \( C' \) bounds a complete collection of meridia. Now remove the 2-handle
to regain $W$. Its complement $H$ is obtained by attaching a 1-handle to $H'$, so $H$ is also a union of handlebodies.

Subcase (a). $E$ is separating. Then $C'$ also contains a complete collection of meridia for $S^3 - W$.

If $E$ is nonseparating, the sides of the 2-handle $E_{\pm}$ lie on the same component $T'$ of $\partial W'$. Let $T$ be the corresponding component of $\partial W$, containing $\partial E$.

Subcase (b). $E_{\pm}$ lie in different components of $T' - C$. Then there is a curve $c' \in C$ that separates $T'$ so that $E_{+}$ and $E_{-}$ lie in different components of $T' - c'$. Since $C' \subset C$ contains a complete collection of meridia $C''$ of $H'$, it follows that $c'$ bounds a separating disk $D$ in $H'$. After sliding the entire contents (handles and curves) of one component of $H' - D$ over the 1-handle dual to $E$, $D$ becomes the cocore of that 1-handle and $C'' \cup c' \subset C$ becomes a complete collection of meridia curves for $H$.

Subcase (c). $E_{\pm}$ lie in the same component of $T' - C$. Then there is a circle $d$ in $T - C$ intersecting $\partial E$ precisely once. If $T$ is not a torus, then banding together $E_{\pm}$ along $d - E$ gives an essential separating disk in $S^3 - W$ with boundary in $T - C$, i.e., Subcase (a). If $T$ is a torus, then $\partial E \cup d$ is a spine of $T$, so $T$ is disjoint from $C$. Consider $Q = \partial W - T$. The image of $H_1(Q - C)$ in $H_1(Q)$ has rank at least $\text{genus}(Q) = \text{genus}(\partial W) - 1$. Hence the image of $H_1(\partial W - C)$ in $H_1(\partial W)$ has rank at least $\text{genus}(Q) + \beta_1(T) = \text{genus}(\partial W) + 1$. But it is a well-known consequence of Poincaré duality that the image of $H_1(\partial W)$ in $H_1(W)$ has rank $= \text{genus}(\partial W)$. Hence there is an $\alpha$ in $H_1(\partial W - C)$ that maps nontrivially to $H_1(\partial W)$ but trivially to $H_1(W)$. Then an element in $H_2(W, \partial W - C)$ mapping to $\alpha$, cannot map trivially to $H_2(W, \partial W)$. This contradicts the hypothesis that $H_2(W, \partial W - C) \to H_2(W, \partial W)$ is trivial. $\square$

Remark. The condition on framings is necessary: Let $T$ be the standard unknotted torus in $S^3$, separating $S^3$ into two solid tori, one with meridian $\mu$, and the other with meridian $\lambda$. Let $W = T \times I$ be a collar so that $\mu \times \{0\}$ and $\lambda \times \{1\}$ are meridia for the solid tori $S^3 - W$. Let $c_1 = \mu \times \{0\}$ and $c_2$ be the curve in $T \times \{1\}$ homologous to $\mu + p\lambda$, $|p| > 1$. Then $c_1 \cup c_2$ is the unlink in $S^3$ and $H_2(W, \partial W - C) = 0$. But since in $T$, $c_1 \cdot c_2 = p \neq \pm 1$, there is no embedding of $W$ in $S^3$ for which $c_1$ and $c_2$ are meridia of distinct complementary solid tori.

To see the difficulty, note that any disk that $c_2$ bounds in $W$ will have a transverse intersection arc with $\partial W$. This arc is a consequence of the mismatch of framings and immediately prevents the construction of $E$ in the proof of the theorem.

References


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