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HANDLEBODY COMPLEMENTS IN THE 3-SPHERE: A REMARK ON A THEOREM OF FOX

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ABSTRACT. Let W be a compact 3-dimensional submanifold of S^3, and C be a collection of disjoint simple closed curves on ∂W. We give necessary and sufficient conditions (one extrinsic, one intrinsic) for W to have an imbedding in S^3 so that $S^3 - W$ is a union of handlebodies, and C contains a complete collection of meridia for these handlebodies.

A theorem of Fox [F] says that any connected compact 3-dimensional submanifold W of S^3 is homeomorphic to the complement of a union of handlebodies in S^3. Call a collection C of simple closed curves in ∂W *proto-meridinal* if there is some embedding of W in S^3 so that the complement of W is a union of handlebodies and C contains a complete collection of meridia for this handlebody. It is natural to ask whether a given collection C of simple closed curves in ∂W is proto-meridinal. This is a difficult question to answer. For example, if ∂W is a torus, either W is a solid torus or only one curve in ∂W is proto-meridinal. The proof is essentially the celebrated solution by Gordon and Luecke [GL] of the knot complement conjecture.

For genus$(\partial W) > 1$ the problem seems intractable. But it does make sense to ask the following question: What conditions on the given embedding of W in S^3 (called *extrinsic* conditions) and what conditions on the pair (W, C) itself (called *intrinsic* conditions) suffice to guarantee that C is proto-meridinal? The general goal is to discover intrinsic conditions that allow extrinsic conditions to be weakened. Here it is shown that a simple intrinsic condition allows one to reduce the extrinsic requirements to a simple condition on C regarded as a link in S^3.

Let C be a collection of disjoint simple closed curves in $\partial W \subset W \subset S^3$. The normal direction to ∂W in S^3 determines (up to orientation) a framing of the normal bundle $\eta(C)$ of C in S^3. Suppose C is the unlink in S^3 and Δ is a collection of disjoint disks that C bounds. Then the normal of C into Δ also determines (up to orientation) a framing on $\eta(C)$. If the framings coincide (up to orientation) then we say that C is a *framed unlink.*

A collection of disjoint simple closed curves in the boundary of a handlebody

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is a *complete collection of meridia* if it bounds a set of disjoint disks whose complementary closure is the 3-ball.

Theorem. *Suppose W is a connected compact 3-dimensional submanifold of S^3 and C is a family of disjoint simple closed curves in ∂W that is a framed unlink in S^3. Then C is proto-meridinal if and only if $H_2(W, \partial W - C) \to H_2(W, \partial W)$ is trivial.*

Proof. The condition on H_2 can be stated geometrically: Any properly embedded surface in W whose boundary is disjoint from C must be separating.

One direction is easy. Suppose $S^3 - W$ is a union H of handlebodies and C contains a complete set of meridia for H. Then any simple closed curve in $\partial W - C$ bounds a disk in H. Hence any properly embedded surface S in W whose boundary lies in $\partial W - C$ can be capped off to give a closed surface in S^3, which must be separating. Then S is separating in W.

For the other direction, we first find a compressing disk E for ∂W in $S^3 - W$ such that ∂E is disjoint from C. Recall that C is the framed unlink. Let Δ denote a disjoint collection of disks bounded by C in S^3, chosen to minimize the number of components of intersection with ∂W. Since the framing of $\eta(C)$, given by the normal into Δ coincides with the framing given by the normal to ∂W in S^3, Δ can be put in general position with respect to ∂W so that $\Delta \cap \partial W$ consists of a set of simple closed curves in Δ, possibly just C. Choose Δ so as to minimize the number of components of $\Delta \cap \partial W$. Then an innermost circle of intersection of ∂W with Δ (or a component of $\partial\Delta$ if $\Delta \cap \partial W = \partial\Delta$) is an essential circle in ∂W that is disjoint from (or can be isotoped off of) $\partial\Delta = C$ and bounds a disk E in $S^3 - \partial W$.

We now proceed by induction on $\beta_1 - \beta_0$, where β_i is the ith betti number of the union of all nonspherical components of ∂W. If $\beta_1 - \beta_0 = 0$, then ∂W consists of spheres and there is nothing to prove. The hypothesis is that $H_2(W, \partial W - C) \to H_2(W, \partial W)$ is trivial. With no loss we may assume all curves of C are essential in ∂W. Consider the following possibilities for E:

Case 1. E lies in W. Since $H_2(W, \partial W - C) \to H_2(W, \partial W)$ is trivial, E is separating, and so decomposes W into the boundary connected sum of two manifolds W_1 and W_2, each with boundary of lower genus than that of W. By inductive assumption, each W_i can be embedded in the 3-sphere so that $S^3 - W_i$ is a union of handlebodies H_i in which some subset C_i of $C \cap \partial W_i$ bounds a complete collection of meridia. Then the connected sum of these spheres along 3-balls bisected by the equatorial disk $E \subset \partial W_i$ gives an embedding of W in S^3 whose complement is the boundary sum of H_1 and H_2, hence a union of handlebodies, in which $C_1 \cup C_2 \subset C$, is a complete collection of meridia.

Case 2. E lies in $S^3 - W$. Let W' be the manifold obtained from W by attaching a 2-handle $\eta(E)$ along E. Let C' be the subcollection of C that remains essential in $\partial W'$. Suppose there were a nonseparating surface S' in W' with $\partial S' \subset \partial W' - C'$. Any element of $C - C'$ is inessential in $\partial W'$ so we may isotope $\partial S'$ to be disjoint from C. By general position we may take $S' \cap \eta(E)$ to be disks parallel to E. Then $S = S' \cap W$ would be a nonseparating surface in W with $\partial S \subset \partial W - C$. We conclude that such a surface S' cannot exist; i.e., $H_2(W', \partial W' - C') \to H_2(W', \partial W')$ is trivial. By induction, there is an embedding of W' in S^3 so that $S^3 - W'$ is a union of handlebodies H' in which C' bounds a complete collection of meridia. Now remove the 2-handle to regain W. Its complement H is obtained by attaching a 1-handle to H', so H is also a union of handlebodies.

Subcase (a). E is separating. Then C' also contains a complete collection of meridia for $S^3 - W$.

If E is nonseparating, the sides of the 2-handle $E_\pm$ lie on the same component T' of $\partial W'$. Let T be the corresponding component of ∂W, containing ∂E.

Subcase (b). $E_\pm$ lie in different components of $T' - C$. Then there is a curve $c' \in C$ that separates T' so that E_+ and E_- lie in different components of $T' - c'$. Since $C' \subset C$ contains a complete collection of meridia C'' of H', it follows that c' bounds a separating disk D in H'. After sliding the entire contents (handles and curves) of one component of $H' - D$ over the 1-handle dual to E, D becomes the cocore of that 1-handle and $C'' \cup c' \subset C$ becomes a complete collection of meridia curves for H.

Subcase (c). $E_\pm$ lie in the same component of $T' - C$. Then there is a circle d in $T - C$ intersecting ∂E precisely once. If T is not a torus, then banding together $E_\pm$ along $d - E$ gives an essential separating disk in $S^3 - W$ with boundary in $T - C$, i.e., Subcase (a). If T is a torus, then $\partial E \cup d$ is a spine of T, so T is disjoint from C. Consider $Q = \partial W - T$. The image of $H_1(Q - C)$ in $H_1(Q)$ has rank at least $\text{genus}(Q) = \text{genus}(\partial W) - 1$. Hence the image of $H_1(\partial W - C)$ in $H_1(\partial W)$ has rank at least $\text{genus}(Q) + \beta_1(T) = \text{genus}(\partial W) + 1$. But it is a well-known consequence of Poincaré duality that the image of $H_1(\partial W)$ in $H_1(W)$ has rank $= \text{genus}(\partial W)$. Hence there is an α in $H_1(\partial W - C)$ that maps nontrivially to $H_1(\partial W)$ but trivially to $H_1(W)$. Then an element in $H_2(W, \partial W - C)$ mapping to α, cannot map trivially to $H_2(W, \partial W)$. This contradicts the hypothesis that $H_2(W, \partial W - C) \to H_2(W, \partial W)$ is trivial. □

Remark. The condition on framings is necessary: Let T be the standard unknotted torus in S^3, separating S^3 into two solid tori, one with meridian μ, and the other with meridian λ. Let $W = T \times I$ be a collar so that $\mu \times \{0\}$ and $\lambda \times \{1\}$ are meridia for the solid tori $S^3 - W$. Let $c_1 = \mu \times \{0\}$ and c_2 be the curve in $T \times \{1\}$ homologous to $\mu + p\lambda$, $|p| > 1$. Then $c_1 \cup c_2$ is the unlink in S^3 and $H_2(W, \partial W - C) = 0$. But since in T, $c_1 \cdot c_2 = p \neq \pm 1$, there is no embedding of W in S^3 for which c_1 and c_2 are meridia of distinct complementary solid tori.

To see the difficulty, note that any disk that c_2 bounds in W will have a transverse intersection arc with ∂W. This arc is a consequence of the mismatch of framings and immediately prevents the construction of E in the proof of the theorem.

References

[F] R. Fox, *On imbeddings of polyhedra in 3-space*, Ann. of Math. (2) **49** (1948), 462–470.

[GL] C. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.

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