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RANGE TRANSFORMATIONS ON A BANACH FUNCTION ALGEBRA. IV

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ABSTRACT. Functions in $\mathrm{Op}(I_D, \mathrm{Re}\, A + L)$ are harmonic on D for a closed subalgebra A of $C_0(Y)$, an ideal I of A and a linear subspace L of finite dimension in $C_{0,R}(Y)$ unless the uniform closure of I is selfadjoint.

INTRODUCTION

Let Y be a locally compact Hausdorff space, and A be a uniformly closed subalgebra of $C_0(Y)$. We have shown in [5, Theorem 2] that functions in $\mathrm{Op}(I_D, \mathrm{Re}\, A)$ for certain ideals I of a closed subalgebra A of $C_0(Y)$, and a plane domain D containing the origin, are harmonic near the origin. In this paper we show that the functions are harmonic on the whole of D for more general range transformations. The proof of Theorem 2 in [5] depends on analysis of the behavior of functions near each point of $Y - \mathrm{Ker}\, I$. In this paper we show that it is enough to analyze near the strong boundary points.

Notations and terminologies are essentially due to [5], while in this paper, $\|\cdot\|_{\infty(K)}$ denotes the uniform norm on a set K, which is simply denoted by $\|\cdot\|_\infty$ in [5].

Lemma 1. *Let X be a compact Hausdorff space, A a closed subalgebra of $\{f \in C(X) : f|K = 0\}$ for a (possibly empty) compact subset K of X which satisfies that for any pair of different points x in $X - K$ and y in X, there exists f in A with $f(x) \neq f(y)$. Suppose that each x in the Choquet boundary* $\mathrm{Ch}(A)$ *for A, with at most finitely many exceptions, has a compact neighborhood G_x such that $A|G_x = C(G_x)$. Then A coincides with $\{f \in C(X) : f|K = 0\}$.*

Proof. We may assume $K \neq X$. If K is empty, then we add ∞ as an isolated point and X' denotes $X \cup \{\infty\}$ and A' denotes $\{f \in C(X') : f(\infty) = 0,\ f|X \in A\}$. If K is not empty, then X' denotes the quotient space of X obtained by identifying points in K, and we denote the point in X' which corresponds to K by ∞. We may suppose that A is a closed subalgebra of $\{f \in C(X') : f(\infty) = 0\}$. We write A' instead of A if we view A as an algebra on X'. In any case A' is a point separating uniformly closed subalgebra of $\{f \in C(X') : f(\infty) = 0\}$.

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Let B denote the function algebra $\{f \in C(X') : f = g + c$ for $g \in A'$ and a constant function c on $X'\}$. If B coincides with $C(X')$, then we see that $A' = \{f \in C(X') : f(\infty) = 0\}$, in particular, $A = \{f \in C(X) : f|K = 0\}$. Thus we will show that $B = C(X')$. First we show that the family of peak sets for A' coincides with the family of peak sets for B which do not contain ∞ . Every peak set for A' is trivially a peak set for B without ∞ , so we prove the inverse implication. Let P be a peak set for B with ∞ off P and g a peaking function for P, that is, g is a function in B with $\|g\|_{\infty(X')} = 1$ such that $g = 1$ on P and $|g| < 1$ off P. Without loss of generality we may suppose that $0 \le |g(\infty)| < \frac{1}{3}$. Put $h = (g - g(\infty))/(1 - g(\infty))$. We see that h is a function in A' with $h = 1$ on P and $\|h\|_\infty \le 2$. For every nonnegative integer n put $G_n = \{x \in X' : |h(x)| \le 1 + 2^{-n}\}$. Then G_n is a compact neighborhood of P with $G_n \supset G_{n+1}$ for every n. Since each $\operatorname{Int} G_n$ includes P, we have

$$\sup\{|g(x)| : x \in X' - G_n\} = \delta_n < 1$$

for every positive integer n. Choose positive integers $m(n)$ with $\delta_n^{m(n)} < 2^{-(n+1)}$. We see that $f_N = \sum_{n=1}^{N} 2^{-n} h g^{m(n)}$ is in A' for every positive integer N since $B \cdot A' \subset A'$ and that f_N converges to a peaking function f for P in A'. We conclude that the two families of peak sets coincide. It follows that $\operatorname{Ch}(B) - \{\infty\} = \operatorname{Ch}(A')$. We may suppose that $\operatorname{Ch}(A) = \operatorname{Ch}(A')$. Let $\{X_\alpha\}$ be the maximal antisymmetric decomposition of X' for B. We see that $\operatorname{Ch}(B|X_\alpha) = \operatorname{Ch}(B) \cap X_\alpha$ for every X_α since X_α is an intersection of peak sets. We show that every X_α is singleton. Suppose not. Choose X_α which is not singleton. Then X_α contains an infinite number of points in $\operatorname{Ch}(B|X_\alpha)$. By the condition on $\operatorname{Ch}(A)$ there is a point x in $\operatorname{Ch}(B|X_\alpha)$ which has a compact neighborhood G_x in X' with $B|G_x \supset A|G_x = C(G_x)$. Thus $X_\alpha \cap G_x$ is an interpolating compact neighborhood of x for $B|X_\alpha$. It follows by Lemma 2.5 in [2] that X_α is a single point, which is a contradiction. Thus we see that every X_α is a finite set, so every X_α is a one point set. We conclude that $B = C(X')$ by Bishop's theorem.

Lemma 2. *Let F be a compact Hausdorff space and Q a compact subset (possibly empty) of F. Let S be a Banach algebra contained in $\{f \in C(F) : f|Q = 0\}$ with the norm $\|\cdot\|_S$ such that S separates a point x in F and a point y in $F - Q$ if $x \ne y$. Suppose that $Q = \{x \in F : f(x) = 0$ for every f in $S\}$. Suppose that $\widetilde{S}^\Lambda$ (with respect to $\|\cdot\|_S$) separates the different points in F_x^Λ for every $x \in F - Q$ and a discrete space Λ with cardinality not less than that of an open base for the topology of Y at x. Let ε be a positive number and h_0 a continuous real valued function on $\{z \in \mathbf{C} : |z| \le \varepsilon\}$ which is not harmonic on any open neighborhood of the origin. Let T be a Banach space with the norm $\|\cdot\|_T$ continuously embedded in $C_R(F)$, that is, the identity map from T into $C_R(F)$ is continuous. Let S_0 be a dense (with respect to the topology induced by $\|\cdot\|_S$) subset of $\{f \in S : \|f\|_S < \delta\}$ for a positive number δ with $\delta < \varepsilon$. Suppose that the function $h_0(f)$ is in T for every $f \in S_0$. Then for every point x in $F - Q$ there exists a compact neighborhood G of x with $G \subset F - Q$ which satisfies $\{f \in T : f|Q = 0\}|G = C_R(G)$.*

Proof. We denote $T_0 = \{f \in T : f|Q = 0\}$ and $T_1 = \{f \in T : f|Q$ is constant$\}$ with the norm $\|\cdot\|_T$ restricted to T_0 and T_1 respectively. We will show that $\mathrm{cl}(\widetilde{T}_0^\Lambda|F_x^\Lambda) = C_R(F_x^\Lambda)$ for a point x in $F - Q$ and a discrete space Λ with cardinality not less than that of an open base for the topology of Y at x. It follows that $T_0|G = C_R(G)$ for a compact neighborhood G of x with $G \subset F - Q$ by (2) of Theorem 1 in [5]. (The theorem is stated for a Banach space included (see [5, p. 89] for the definition) in $C_0(Y)$ or $C_{0,R}(Y)$ in [5]. Every Banach space continuously embedded in $C_0(Y)$ or $C_{0,R}(Y)$ has a norm equivalent to that of a Banach space included in $C_0(Y)$ or $C_{0,R}(Y)$. So the theorem is also true for Banach spaces continuouly embedded in $C_0(Y)$ or $C_{0,R}(Y)$.) Take a point x in $F-Q$. Then there is f in S with $f(x) = 1$. For some η with $0 < \eta < \varepsilon/(2\|f\|_S)$, and some smoothing operator $\sigma_\eta(w)$ of class C^∞ supported in $\{z \in \mathbf{C} : |z| < \eta\}$, we have $\Delta_1(h_\eta(0, 1)) \neq 0$ (see [3, p. 566; 4, pp. 634, 635; 5, p. 115], where

$$h_\eta(z_1, z_2) = \iint h_0(z_1 - z_2 w)\sigma_\eta(w)\,dx\,dy.$$

Choose a discrete space Λ with cardinality not less than that of an open base for the topology of Y at x. Let $\tilde{g}$ be a function in $\widetilde{S}^\Lambda$. For a complex number β with sufficiently small absolute value and a complex number w with $|w| < \eta$, $h_0(\tilde{g}\langle f\rangle^2\beta - \langle f\rangle w)$ is in $\mathrm{cl}\,\widetilde{T}_1^\Lambda$ since S_0 is dense (with respect to the topology induced by $\|\cdot\|_S$) in $\{f \in S : \|f\|_S \leq \delta\}$ and $\|\cdot\|_{\infty(\widetilde{F}^\Lambda)} \leq \|\cdot\|_{\widetilde{S}^\Lambda}$. Thus we have by Lemma 5 in [4], that $|\tilde{g}|^2\Delta_1(h_\eta(0, \langle f\rangle) \cdot |\langle f\rangle|^4) \in \{\tilde{\psi} \in \mathrm{cl}\,\widetilde{T}_1^\Lambda : \tilde{\psi}|\widetilde{Q}^\Lambda = 0\}$ since $\tilde{g} = 0$ on $\widetilde{Q}^\Lambda$. We show that $\mathrm{cl}\,\widetilde{T}_0^\Lambda = \{\tilde{\psi} \in \mathrm{cl}\,\widetilde{T}_1^\Lambda : \tilde{\psi}|\widetilde{Q}^\Lambda = 0\}$. When $T_1 = T_0$ it is trivial. Suppose that $T_1 - T_0 \neq \phi$. Let $\tilde{\psi} = \langle\psi_\lambda\rangle$ be in $\mathrm{cl}\,\widetilde{T}_1^\Lambda$ with $\tilde{\psi}|\widetilde{Q}^\Lambda = 0$. For every $\xi > 0$ there is $\langle\psi_\lambda^{(\xi)}\rangle$ in $\widetilde{T}_1^\Lambda$ such that

$$\|\langle\psi_\lambda\rangle - \langle\psi_\lambda^{(\xi)}\rangle\|_{\infty(\widetilde{F}^\Lambda)} < \xi.$$

Let ψ be a function in $T_1 - T_0$. We may assume that $\psi|Q = 1$. So

$$\langle\psi_\lambda^{(\xi)} - \psi_\lambda^{(\xi)}(Q)\psi\rangle \in \widetilde{T}_0^\Lambda$$

and

$$\|\langle\psi_\lambda\rangle - (\langle\psi_\lambda^{(\xi)} - \psi_\lambda^{(\xi)}(Q)\psi\rangle)\|_{\infty(\widetilde{F}^\Lambda)} < \xi(1 + \|\psi\|_{\infty(F)}),$$

since $\psi_\lambda^{(\xi)}(Q)$ is a constant with $|\psi_\lambda^{(\xi)}(Q)| \leq \|\langle\psi_\lambda\rangle - \langle\psi_\lambda^{(\xi)}\rangle\|_{\infty(\widetilde{F}^\Lambda)}$. Thus the conclusion follows. We see that $|\tilde{g}|^2\Delta_1(h_\eta(0, \langle f\rangle) \cdot |\langle f\rangle|^4)$ is in $\mathrm{cl}\,\widetilde{T}_0^\Lambda$, so we have $|\tilde{g}|^2\Delta_1(h_\eta(0, 1))|F_x^\Lambda \in (\mathrm{cl}\,\widetilde{T}_0^\Lambda)|F_x^\Lambda \subset \mathrm{cl}(\widetilde{T}_0^\Lambda|F_x^\Lambda)$, since $\langle f\rangle = 1$ on F_x^Λ by Lemma 4 in [5]. So $|\tilde{g}|^2|F_x^\Lambda \in \mathrm{cl}(\widetilde{T}_0^\Lambda|F_x^\Lambda)$ since $\Delta_1(h_\eta(0, 1)) \neq 0$. It follows that the algebra generated by $|\tilde{g}|^2|F_x^\Lambda$ for $\tilde{g} \in \widetilde{S}^\Lambda$ is contained in $\mathrm{cl}(\widetilde{T}_0^\Lambda|F_x^\Lambda)$ since $\widetilde{S}^\Lambda$ is an algebra. Let p and q be different points in F_x^Λ. There is $\tilde{u}$ in $\widetilde{S}^\Lambda$ with $\tilde{u}(p) \neq \tilde{u}(q)$ and there is v in S with $v(x) = \tilde{u}(p)$. So $\tilde{u} - \langle v\rangle$ is in $\widetilde{S}^\Lambda$ and $|\tilde{u} - \langle v\rangle|^2$ separates p and q since $\langle v\rangle(p) = \langle v\rangle(q) = v(x)$ by Lemma 4 in [5]. It follows by the Stone-Weierstrass theorem that $\mathrm{cl}(\widetilde{T}_0^\Lambda|F_x^\Lambda) = C_R(F_x^\Lambda)$.

Lemma 3. *Let E be a complex (resp. real) Banach space continuously embedded in $C(X)$ (resp. $C_R(X)$), and let L be a finite-dimensional complex (resp. real) subspace of $C(X)$ (resp. $C_R(X)$) such that $E \cap L = \{0\}$. Let Λ be a discrete space. Then $E + L$ is a Banach space with respect to the norm defined by*

$$\|u + v\|_{E+L} = \|u\|_E + \|v\|_\infty,$$

where $\|\cdot\|_E$ *is the norm on* E. *Suppose that* x *is a point in* X *with* $E_x \neq E$. *Then*

$$(E+L)^{\sim\Lambda}|F_x^\Lambda = \widetilde{E}^\Lambda|F_x^\Lambda.$$

Proof. It is trivial that $\|\cdot\|_{E+L}$ is a complete norm since L is finite dimensional. $(E+L)^{\sim\Lambda} \supset \widetilde{E}^\Lambda$ is also trivial, so $(E+L)^{\sim\Lambda}|F_x^\Lambda \supset \widetilde{E}^\Lambda|F_x^\Lambda$. We prove the inverse inclusion. Suppose that $\{k_1, \dots, k_n\}$ is a base for L. Let $\langle u_\lambda + v_\lambda\rangle \in (E+L)^{\sim\Lambda}$, where $u_\lambda \in E$ and $v_\lambda = \sum_{i=1}^n a_i^{(\lambda)} k_i$. Then $\langle u_\lambda\rangle \in \widetilde{E}^\Lambda$ and $\sup\|v_\lambda\|_{\infty(X)} < \infty$ by the definition of $\|\cdot\|_{E+L}$. Let u be a function in E with $u(x) = 1$. Since the linear functional defined by $\sum_{i=1}^n a_i k_i \to a_j$, is continuous for every j, we see that $\langle u_\lambda\rangle + \sum_{i=1}^n \langle a_i^{(\lambda)} k_i(x) u\rangle$ is in $\widetilde{E}^\Lambda$. We see that

$$\langle u_\lambda\rangle|F_x^\Lambda + \sum_{i=1}^n \langle a_i^{(\lambda)} k_i(x) u\rangle|F_x^\Lambda = \langle u_\lambda\rangle|F_x^\Lambda + \sum_{i=1}^n \langle a_i^{(\lambda)} k_i\rangle|F_x^\Lambda$$
$$= \langle u_\lambda\rangle|F_x^\Lambda + \left\langle \sum_{i=1}^n a_i^{(\lambda)} k_i \right\rangle |F_x^\Lambda = \langle u_\lambda + v_\lambda\rangle|F_x^\Lambda$$

by Lemma 4 of [5].

Lemma 4. *Let* E *be an ultraseparating complex* (*resp. real*) *Banach space continuously embedded in* $C(X)$ (*resp.* $C_R(X)$) *for a compact Hausdorff space* X, *i.e.,* $\|f\|_\infty \le M\|f\|_E$ *for every* f *in* E. *Let* x *be a point in* X. *Let* Λ *and* Λ' *be infinite discrete spaces. If* $p \in F_x^\Lambda$ *and* $q \in F_x^\Lambda - [\{x\} \times \Lambda]$ *are different points, then* $\widetilde{E}_x^\Lambda$ *separates* p *and* q. *If* $q \in F_x^\Lambda - [\{x\} \times \Lambda]$, *then* $(\widetilde{E}_x^\Lambda)^{\sim\Lambda'}$ *separates the different points in* $F_q^{\Lambda'}$.

Proof. If $p \in [\{x\}\times\Lambda]$ and $q \in F_x^\Lambda - [\{x\}\times\Lambda]$, then there is $\tilde{g} \in \widetilde{E}_x^\Lambda$ such that $\tilde{g}(q) \neq \tilde{g}(p) = 0$ by Proposition 2 in [5]. Suppose that p and q are different points in $F_x^\Lambda - [\{x\} \times \Lambda]$. Since E is ultraseparating, there is $\tilde{f} = \langle f_\lambda\rangle$ in $\widetilde{E}^\Lambda$ with $\tilde{f}(p) \neq \tilde{f}(q)$. Put $h_\lambda = f_\lambda - f_\lambda(x)\cdot u$, where $u \in E$ with $u(x) = 1$. (Such a function u exists since $\widetilde{E}^\Lambda$ separates (x, λ) and (x, λ') for different λ and λ'.) So $\langle h_\lambda\rangle \in \widetilde{E}_x^\Lambda$. Take $\tilde{g} \in \widetilde{E}_x^\Lambda$ with $\tilde{g}(p) \neq 0$. Then $\langle h_\lambda\rangle$ or $\tilde{g}$ or $\langle f_\lambda(x)\rangle\cdot\tilde{g}$ separates p and q. Let q be a point in $F_x^\Lambda - [\{x\}\times\Lambda]$. By Corollary 1 of [5], $(\widetilde{E}^\Lambda)^{\sim\Lambda'}$ separates the different points of $(\widetilde{X}^\Lambda)^{\sim\Lambda'}$, in particular, if α and β are different points in $F_q^{\Lambda'}$, then there is $\tilde{\tilde{f}} = \langle\langle f_{\lambda,\lambda'}\rangle\rangle \in (\widetilde{E}^\Lambda)^{\sim\Lambda'}$ with $\tilde{\tilde{f}}(\alpha) \neq \tilde{\tilde{f}}(\beta)$. Put $h_{\lambda,\lambda'} = f_{\lambda,\lambda'} - f_{\lambda,\lambda'}(x)\cdot u$, so $\langle\langle h_{\lambda,\lambda'}\rangle\rangle \in (\widetilde{E}_x^\Lambda)^{\sim\Lambda'}$. Then $\langle\langle h_{\lambda,\lambda'}\rangle\rangle$ or $\langle\langle f_{\lambda,\lambda'}(x)\rangle\rangle\cdot\tilde{g}\rangle$ separates α and β for $\tilde{g} \in \widetilde{E}_x^\Lambda$ with $\tilde{g}(q) \neq 0$.

Theorem. *Let* A *be a uniformly closed subalgebra of* $C_0(Y)$ *for a locally compact Hausdorff space* Y, *and* I *be a subalgebra of* A *such that there are a finite number of subalgebras* $I_0, I_1, \dots, I_n$ *of* A *which satisfy the condition that* I_k *is an ideal of* $\operatorname{cl} I_{k-1}$ *for every* $k = 1, 2, \dots, n$, *where* $I_0 = A$ *and* $I_n = I$. *Let* D *be a plane domain containing the origin. Let* L *be a linear subspace of finite dimension in* $C_{0,R}(Y)$. *Suppose that* $\operatorname{Op}(I_D, \operatorname{Re} A + L)$ *contains a function which is not harmonic on* D. *Then* $I|K$ *is uniformly closed and selfadjoint for every compact subset* K *of* $Y - \operatorname{Ker} I$ *and* $\operatorname{cl} I$ *is selfadjoint.*

Remark. If I is an ideal of A, then the condition on I and A is satisfied with $n = 1$. An ideal of an ideal need not be an ideal. Let $A(\overline{\Delta})$ be the disk algebra on the closed unit disk $\overline{\Delta}$. Let $I = \{f \in A(\overline{\Delta}) : f(0) = f'(0) = 0\}$ and $J = \{f \in I : f'''(0) = 0\}$. Then I is an ideal of $A(\overline{\Delta})$ and J is an ideal of I while J does not satisfy the condition $J \cdot A(\overline{\Delta}) \subset J$.

Proof of Theorem. The notations $\overline{Y}$, ∞, $\overline{Y}_1$, p, $\overline{Y}_0$, and I' are the same as in the proof of Theorem 2 in [5]. If Y is not compact, then $\overline{Y}$ denotes the one point compactification of Y and ∞ denotes the point in $\overline{Y} - Y$. If Y is compact, then we add ∞ as an isolated point and $\overline{Y}$ denotes $Y \cup \{\infty\}$. We may suppose that A is a closed subalgebera of $C(\overline{Y})$ such that $f(\infty) = 0$ for every f in A. Let $\overline{Y}_1$ be the quotient space obtained by identifying the points in $\overline{Y}$ which cannot be separated by A. Let $\overline{Y}_0$ be the quotient space obtained by identifying the points in $\overline{Y}$ which cannot be separated by I. Let p be the point in $\overline{Y}_0$ which corresponds to the equivalence class in $\overline{Y}$ containing ∞. We may suppose that $\overline{Y}_0$ is the quotient space obtained by identifying points in $\overline{Y}_1$ which cannot be separated by I and that p corresponds to $\mathrm{Ker}\, I$. We may also suppose that each point in $\overline{Y}_0 - \{p\}$ corresponds to a point in $\overline{Y}_1 - \mathrm{Ker}\, I$, that is, we may suppose that $\overline{Y}_0 - \{p\} = \overline{Y}_1 - \mathrm{Ker}\, I$. Let $I' = \mathrm{cl}\, I + C$ be the sum of the uniform closure of I and the space of constant functions C. Then I' is a function algebra on $\overline{Y}_0$. Let $\mathrm{Ch}(I')$ be the Choquet boundary for I'. We consider two cases. They are different from those of the proof of Theorem 2 in [5] : (1) There is no accumulation point of $\mathrm{Ch}(I')$ which is a point in $\mathrm{Ch}(I')$, or p is the only accumulation point of $\mathrm{Ch}(I')$ which is a point in $\mathrm{Ch}(I')$. (2) There is an accumulation point of $\mathrm{Ch}(I')$ which is also a point in $\mathrm{Ch}(I')$ and is not p.

Case (1). Let Γ be the Shilov boundary for I'. We may suppose that I' is a function algebra on Γ. We show that $\{x\}$ is itself a compact neighborhood of x for every x in $\mathrm{Ch}(I') - \{p\}$. Since x is not an accumulation point of $\mathrm{Ch}(I')$, there is an open neighborhood U_x in Γ of x with $U_x \cap \mathrm{Ch}(I') = \{x\}$. If a point y is in $U_x - \{x\}$, then y is in Γ. So there is an open neighborhood V_y in Γ of y such that $V_y \subset U_x$ and x is not contained in V_y. Since $V_y \cap \mathrm{Ch}(I') \subset U_x \cap \mathrm{Ch}(I') = \{x\}$ and x is not contained in V_y we have $V_y \cap \mathrm{Ch}(I') = \varnothing$, which is a contradiction since $y \in \Gamma$ and $\mathrm{Ch}(I')$ is dense in Γ. We conclude that $U_x = \{x\}$, so $\{x\}$ is a compact neighborhood of x. We see by Lemma 1 that $I'|\Gamma = C(\Gamma)$ since $I'|U_x = C(U_x)$ for every x in $\mathrm{Ch}(I') - \{p\}$. It follows that $\overline{Y}_0 = \Gamma$ and $I' = C(\overline{Y}_0)$. The rest of the proof is the same as in case (1) of the proof of Theorem 2 in [5].

Case (2). Choose a point q in $\mathrm{Ch}(I')$ which is also an accumulation point of $\mathrm{Ch}(I')$ other than p. Suppose that h is a function in $\mathrm{Op}(I_D, \mathrm{Re}\, A + L)$ which is not harmonic on D. First we show that h is continuous on D. Suppose not. There is a point a in D such that h is not continuous at a. There is a function k in I such that $k(q) = d > 0$. $\|k\|_\infty \le 1$. By the proof of Lemma 1, $\mathrm{Ch}(I') - \{p\} = \mathrm{Ch}(\mathrm{cl}\, I)$, so q is a point in $\mathrm{Ch}(\mathrm{cl}\, I)$. Thus there is a function u in $\mathrm{cl}\, I$ with $u(q) = 1$, $\|u\|_\infty \le 1$. Choose an analytic function H defined on the open unit disk Δ with range in D, such that $H(0) = 0$ and $a \in H(\Delta)$. Such a function exists since D is connected. Let $re^{i\theta}$ be a point in Δ with $H(re^{i\theta}) = a$. So $H(re^{i\theta}u)$ is a function in $\mathrm{cl}\, I$ such that $H(re^{i\theta}u)(q) = a$ and $H(re^{i\theta}u)(\overline{Y}_0)$ is a compact subset of D. In particular, there is a positive ε with $\bigcup_{y \in \overline{Y}_0} \{z \in C : |z - H(re^{i\theta}u)(y)| \le 2\varepsilon\} \subset D$. There is a function ψ in I with $\psi(q) = 1$. Choose a function ψ_1 in I with $\|\psi_1 - H(re^{i\theta}u)\|_{\infty(\overline{Y}_0)} < \varepsilon/(2\|\psi\|_{\infty(\overline{Y}_0)})$. Put

$$\psi_q = \psi_1 - (\psi_1(q) - a)\psi,$$

so $\psi_q(q) = a$. For a point y in $\overline{Y}_0$, suppose that z is a complex number with $|z - \psi_q(y)| \leq \varepsilon$. Then we have $|z - H(re^{i\theta}u)(y)| \leq 2\varepsilon$. We conclude that

$$\bigcup_{y\in\overline{Y}_0} \{z \in C : |z - \psi_q(y)| \leq \varepsilon\} \subset D.$$

As in the proof of Theorem 2 in [5] we can choose a sequence $\{q_n\}$ in $\mathrm{Ch}(I')$ with an accumulation point q_0'' and a function f in $\mathrm{cl}\, I$ such that $h(fk + \psi_q) \in \mathrm{Re}\, A + L$ and that $h(fk + \psi_q)(q_n)$ does not converge to $h(fk + \psi_q)(q_0'')$, which is a contradiction. Thus we may assume that h is continuous on D in case (2). Choose a point a in D such that h is not harmonic on any open neighborhood of a. We show that for every x in $\mathrm{Ch}(\mathrm{cl}\, I)$ there exists a compact neighborhood G_x of x with $\mathrm{cl}\, I|G_x = C(G_x)$. If so, $\mathrm{cl}\, I$ is selfadjoint by Lemma 1. Let x be a point in $\mathrm{Ch}(\mathrm{cl}\, I)$. Then there is a function ϕ in I with $\phi(x) = a$ and $\phi(\overline{Y}_0) \subset D$ in the same way as above. Let f_1 be a function in I with $f_1(x) = 1$. Put $f_0 = f_1^{n+1}$. Since $\mathrm{cl}\, I_k$ is an ideal of $\mathrm{cl}\, I_{k-1}$ by the condition on I_k we have $f_1^k \cdot A \subset \mathrm{cl}\, I_k$. Thus $f_0 \cdot A \subset I$ and $f_0 \cdot A$ is a Banach algebra contained in $C(\overline{Y}_0)$ with respect to the norm defined by

$$\|u\|_{f_0 \cdot A} = \inf\{\|f_0\|_\infty \cdot \|g\|_\infty : g \in A,\, u = f_0 g\}$$

for u in $f_0 \cdot A$. Without loss of generality we may assume $(\mathrm{Re}\, A) \cap L = \{0\}$. $\mathrm{Re}\, A + L$ is a Banach space with respect to the norm defined by

$$\|u + v\|_{\mathrm{Re}\, A+L} = \|u\|_{\mathrm{Re}\, A} + \|v\|_\infty$$

for every $u \in \mathrm{Re}\, A$ and $v \in L$ by Lemma 3. We show that $f_0 \cdot A$ is ultraseparating near x. Let ε be a positive number such that $d(\phi(\overline{Y}_0), D^c) > \varepsilon$, where $d(\cdot, \cdot)$ is the usual Euclidian distance and D^c is the complement of D in the complex plane C. Since h is not harmonic near a, we see that

$$|\Delta_1(h_\eta(z, z'))| \geq (1/2)|\Delta_1(h_\eta(a, 1))| \neq 0$$

on $\{(z, z') \in C^2 : |z - a| < \varepsilon'', |z' - 1| < \varepsilon''\}$ for a suitably chosen η with $\varepsilon/(2\|f_0\|_\infty) > \eta > 0$ and a suitably chosen smoothing operator σ_η and an $\varepsilon > \varepsilon'' > 0$, where

$$h_\eta(z, z') = \iint h(z - z'w)\sigma_\eta(w)\,dx\,dy$$

in the same way as in [4, pp. 634, 635]. Since L is finite dimensional, we have $\mathrm{cl}(\mathrm{Re}\, A + L) = (\mathrm{cl}\,\mathrm{Re}\, A) + L$. So we see that

$$h_\eta(\phi + g_1 f_0^2 t, f_0) \in (\mathrm{cl}\,\mathrm{Re}\, A) + L$$

for $g_1 \in A + C$ and a complex number t with sufficiently small absolute value. Thus by Lemma 5 in [4]

$$|g_1|^2 |\Delta_1(h_\eta(\phi, f_0) \cdot |f_0|^4) \in (\mathrm{cl}\,\mathrm{Re}\, A) + L.$$

So by the Stone-Weierstrass theorem we see that

$$C(\overline{Y}_1)\cdot\Delta_1(h_\eta(\phi, f_0)\cdot|f_0|^4)\subset(\mathrm{cl}\,\mathrm{Re}\,A)+L.$$

Let G_x be a compact neighborhood of x such that

$$G_x=\{y\in\overline{Y}_0:|\phi(y)-a|\le\varepsilon''/2,\ |f_0(y)-1|\le\varepsilon''/2|\},$$

so $\Delta_1(h_\eta(\phi, f_0)\cdot|f_0|^4)$ never take zero on G_x. Thus we have

$$C(G_x)=(\mathrm{cl}\,\mathrm{Re}\,A+L)|G_x.$$

Let Λ be a discrete space with cardinality not less than that of an open base for the topology of Y at x. By Lemma 3 we see that $C(F_x^\Lambda)=(\mathrm{cl}\,\mathrm{Re}\,A)^{\sim\Lambda}|G_x$. Thus for different points p and q in F_x^Λ there is a function $\langle u_\lambda\rangle$ in $(\mathrm{cl}\,\mathrm{Re}\,A)^{\sim\Lambda}$ with $\langle u_\lambda\rangle(p)\ne\langle u_\lambda\rangle(q)$, in particular, we may suppose that $u_\lambda\in\mathrm{Re}\,A$. Then there exists $\nu_\lambda\in\mathrm{Re}A$ for every λ such that $u_\lambda+i\nu_\lambda\in A$, so $\exp(u_\lambda+i\nu_\lambda)\in A$ and since $\|\exp(u_\lambda+i\nu_\lambda)\|_{\infty(\overline{Y}_1)}=\|\exp u_\lambda\|_{\infty(\overline{Y}_1)}$ we see that $\langle\exp(u_\lambda+i\nu_\lambda)\rangle\in\widetilde{A}^\Lambda$. By Lemma 4 in [5] we see

$$\langle f_0\cdot\exp(u_\lambda+i\nu_\lambda)\rangle=\langle f_0\rangle\cdot\langle\exp(u_\lambda+i\nu_\lambda)\rangle=\langle\exp(u_\lambda+i\nu_\lambda)\rangle$$

on F_x^Λ. By the definition of $\|\cdot\|_{f_0\cdot A}$ we have

$$\langle f_0\cdot\exp(u_\lambda+i\nu_\lambda)\rangle\in(f_0\cdot A)^{\sim\Lambda}$$

and it separates p and q. So by Theorem 1(1) of [5], $f_0\cdot A$ is ultraseparating near x. Put $B=\{g\in f_0\cdot A:g(x)=0,\ \|g\|_{f_0\cdot A}\le\varepsilon\}$. In the same way as in the proof of Lemma 1.2 in [4] (cf. [5, Lemma 11]) we see the following.

There are positive integers n_0 and a real number ε' with $0<\varepsilon'<\varepsilon$, and a function g_0 in B such that

$$\{g\in f_0\cdot A:g(x)=0,\ \|g-g_0\|_{f_0\cdot A}<\varepsilon'\}\subset B,$$

and there is a dense (with respect to the topology induced by the norm $\|\cdot\|_{f_0\cdot A}$) subset U in $\{g\in f_0\cdot A:g(x)=0,\ \|g-g_0\|_{f_0\cdot A}<\varepsilon'\}$ which satisfies that for every g in U we have

$$h(g+\phi)\in\mathrm{Re}\,A+L\quad\textit{and}\quad\|h(g+\phi)\|_{\mathrm{Re}A+L}<n_0.$$

Let $U_0=\{g\in f_0\cdot A:g+g_0\in U\}$. Then U_0 is dense (with respect to the topology induced by $\|\cdot\|_{f_0\cdot A}$) in $\{g\in f_0\cdot A:g(x)=0,\ \|g\|_{f_0\cdot A}\le\varepsilon'\}$. Put $F=F_x^\Lambda$, $S=(f_0\cdot A_x)^{\sim\Lambda}|F_x^\Lambda$, $S_0=\widetilde{U}_0^\Lambda|F_x^\Lambda$, $Q=[\{x\}\times\Lambda]$, $h_0(z)=h(z+a)$, $\delta=\varepsilon'$, and $T=\mathrm{Re}\,\widetilde{A}^\Lambda|F_x^\Lambda$. Then the conditions of Lemma 2 are satisfied. We check this. F is trivially a compact Hausdorff space and Q is a compact subset of F, and S is a Banach algebra contained in $\{\tilde{f}\in C(F):\tilde{f}|Q=0\}$. We see that the separation and ultraseparation conditions on S are satisfied by Lemma 4. We see that $Q=\{p\in F:\tilde{f}(p)=0$ for every $\tilde{f}$ in $S\}$ by Proposition 2 in [5]. Let $\langle g_\lambda\rangle\in\widetilde{U}_0^\Lambda$. Then $h(\langle g_\lambda\rangle+\langle g_0\rangle+\langle\phi\rangle)$ is in $(\mathrm{Re}\,A+L)^{\sim\Lambda}$ by the definition of U_0. Thus we see that $h(\tilde{g}+a)$ is in $\mathrm{Re}\,\widetilde{A}^\Lambda|F_x^\Lambda$ for every $\tilde{g}$ in $\widetilde{U}_0^\Lambda|F_x^\Lambda$ by Lemma 4 in [5] and Lemma 3. Equivalently $h_0(\tilde{g})\in T$ for every $\tilde{g}\in S_0$. Thus by Lemma 2 we see that for every point p in $F_x^\Lambda-[\{x\}\times\Lambda]$ there is a compact neighborhood O_p of p in F_x^Λ with $O_p\subset F_x^\Lambda-[\{x\}\times\Lambda]$ such that

$$\{\tilde{u}\in\mathrm{Re}\,\widetilde{A}^\Lambda|F_x^\Lambda:\tilde{u}|[\{x\}\times\Lambda]=0\}|O_p=C_R(O_p).$$

We will see that

$$\{\tilde{u} \in \operatorname{Re} \widetilde{A}^{\Lambda}|F_x^{\Lambda} : \tilde{u}|[\{x\} \times \Lambda] = 0\} = \operatorname{Re} \widetilde{A}_x^{\Lambda}|F_x^{\Lambda}.$$

$\{\tilde{u} \in \operatorname{Re} \widetilde{A}^{\Lambda}|F_x^{\Lambda} : \tilde{u}|[\{x\} \times \Lambda] = 0\} \supset \operatorname{Re} \widetilde{A}_x^{\Lambda}|F_x^{\Lambda}$ is trivial. We show the inverse inclusion. If $\tilde{u} \in \operatorname{Re} \widetilde{A}^{\Lambda}|F_x^{\Lambda}$ with $\tilde{u}|[\{x\} \times \Lambda] = 0$, then there is $\langle u_\lambda + i\nu_\lambda\rangle$ in $\widetilde{A}^{\Lambda}$ such that $\langle u_\lambda\rangle|F_x^{\Lambda} = \tilde{u}$. Put $\langle u_\lambda + i\nu_\lambda\rangle - \langle i\nu_\lambda(x) \cdot f_0\rangle \in \widetilde{A}^{\Lambda}$. Since $u_\lambda + i\nu_\lambda - i\nu_\lambda(x) \cdot f_0$ is in A_x, it follows that $\langle u_\lambda + i\nu_\lambda\rangle - \langle i\nu_\lambda(x) \cdot f_0\rangle \in \widetilde{A}_x^{\Lambda}$ and $\operatorname{Re}(\langle u_\lambda + i\nu_\lambda\rangle - \langle i\nu_\lambda(x) \cdot f_0\rangle)|F_x^{\Lambda} = \tilde{u}$ by Lemma 4 in [5]. Thus we have

$$(\operatorname{Re} \widetilde{A}_x^{\Lambda})|O_p = C_R(O_p).$$

We may suppose that $\operatorname{cl}(\widetilde{A}_x^{\Lambda}|F_x^{\Lambda}) + C$ is a function algebra on ${}_0F_x^{\Lambda}$ (${}_0F_x^{\Lambda}$ is the quotient space of F_x^{Λ} obtained by identifying the points in $[\{x\} \times \Lambda]$) by Proposition 2 in [5] since A is ultraseparating near x. By a theorem of Hoffman and Wermer, and Bernard [1, 6] on the uniformly closed real part of a Banach function algebra we see that

$$(\operatorname{cl}(\widetilde{A}_x^{\Lambda}|F_x^{\Lambda}) + C)|O_p = C(O_p),$$

so by Corollary 2.13 in [2] we see that

$$\operatorname{cl}(\widetilde{A}_x^{\Lambda}|F_x^{\Lambda}) + C = C({}_0F_x^{\Lambda}).$$

It follows that

$$\operatorname{cl}(\widetilde{A}^{\Lambda}|F_x^{\Lambda}) = C(F_x^{\Lambda}).$$

We see that $A|G = C(G)$ for a compact neighborhood G of x by Theorem 1 (2) in [5]. It follows that $I|G' = C(G')$ for a compact neighborhood G' of x in $\overline{Y}_0 - \{p\}$. The results follow.

As in Corollary 1.1 in [4] we see the following.

Corollary. *Let A and I and L be the same as in the theorem. Let S be an interval of the real line. Suppose that one of the following holds.*

(1) $\operatorname{Op}(I_D, A)$ contains a nonanalytic function on D.

(2) $\operatorname{Op}((\operatorname{Re} I)_S, \operatorname{Re} A + L)$ contains a nonaffine function on S.

Then $I|K$ is uniformly closed and selfadjoint for every compact subset K of $Y - \operatorname{Ker} I$ and $\operatorname{cl} I$ is selfadjoint.

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