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A NOTE ON A COMMON FIXED POINT THEOREM OF BRODSKII AND MILMAN AND A LEMMA OF DAY

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ABSTRACT. We should show that we may use a lemma of Day to prove, among others, a generalization of a common fixed point theorem of Brodskii and Milman when restricted to normed linear spaces that are uniformly convex in every direction.

INTRODUCTION

Throughout this note N will denote a normed linear space with dual N^*. The norms in N and N^* are denoted by $\|\cdot\|$.

We denote by I the identity mapping of N; by $T: D(T) \subseteq N \to N$ a mapping T with domain $D(T) \subseteq N$ into N and with range of T, $R(T) = \{T(u): u \in D(T)\}$; and for scalars r and s, by $(rI + sT)$ the mapping with domain $D(T)$ such that $(rI + sT)u = rIu + sTu$ for $u \in D(T)$, where $Tu = T(u)$. For scalar r, subsets $R, S \subseteq N$, rS is the set of all ry with $y \in S$ and $R+S$ denotes the set of all $x+y$ with $x \in R$ and $y \in S$. We also write $y+S$ for $\{y\}+S$.

Let $T: D(T) \subseteq N \to N$. T is said to be nonexpansive if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in D(T)$; noncontractive if $\|Tu - Tv\| \geq \|u - v\|$ for all $u, v \in D(T)$; pseudocontractive if $\|((1+r)I-rT)u-((1+r)I-rT)v\| \geq \|u-v\|$ for all $u, v \in D(T)$ and all $r > 0$; accretive if $\|(I + rT)u - (I + rT)v\| \geq \|u - v\|$ for all $u, v \in D(T)$ and all $r > 0$.

Remark 1. It is known that a single-valued nonexpansive map is pseudocontractive.

The importance of pseudocontractive mappings is established by the following characterization in Browder [2]:

> $T: D(T) \subseteq N \to N$ is pseudocontractive if and only if $I - T$ is accretive.

In Kato [7] a multiple-valued mapping $T: D(T) \subseteq N \to N$, that is, Tu is a subset of N for each $u \in D(T)$, is called accretive if for each $r > 0$ and $u, v \in D(T)$,

$$\|x - y\| \leq \|u - v\| \quad \text{whenever } x \in (I + rT)u,\ y \in (I + rT)v.$$

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If $T\colon D(T) \subseteq N \to N$ is multiple-valued and we define T to be pseudocontractive if for each $r > 0$ and $u, v \in D(T)$,

$$\|x - y\| \geq \|u - v\| \quad \text{whenever } x \in ((1+r)I - rT)u,\ y \in ((1+r)I - rT)v,$$

then Browder's characterization of single-valued pseudocontractive mapping still holds for a multiple-valued mapping.

Proposition. *Let $T\colon D(T) \subseteq N \to N$ be multiple-valued. T is pseudocontractive if and only if $I - T$ is accretive.*

Proof. Given $r > 0$; $u, v \in D(T)$; $w \in Tu$, and $z \in Tv$, we have

$$\|((1+r)u - rw) - ((1+r)v - rz)\| = \|(u + r(u - w)) - (v + r(v - z))\|$$

from which the truth of the proposition is easily deduced.

If T is multiple-valued and pseudocontractive, then $((1+r)I - rT)u$ are disjoint for different u and all $r > 0$, so we can define, for each $r > 0$, a single-valued mapping $J_r = ((1+r)I - rT)^{-1}$, with $D(J_r) = R((1+r)I - rT)$ and $R(J_r) = D(T)$, by $J_r x = u$ if and only if $x \in ((1+r)I - rT)u$.

Lemma 1. *For each $r > 0$,*

(i) *J_r is nonexpansive;*
(ii) *J_r and T have the same fixed point(s).*

Proof. (i) This follows directly from the definitions of pseudocontractive mapping and J_r.

(ii) From the definition of J_r, we have

$$J_r u = u \quad \text{if and only if} \quad u \in ((1+r)I - rT)u.$$

Since $u \in Tu$ if and only if $u \in ((1+r)I - rT)u$, (ii) follows.

Remark 2. The definition of a multiple-valued accretive mapping that we use here is that of Kato [7], where it is shown that if N is a real Banach space then T is accretive if and only if for each $u, v \in D(T)$ and each $x \in Tu$ and $y \in Tv$, there exists $f \in F(u - v)$ such that $(x - y, f) \geq 0$. Here F is the duality map of N into N^*; it is by definition the unique multiple-valued mapping from N into N^* with domain $D(F) = N$ such that $f \in Fx$ if and only if $(x, f) = \|x\|^2 = \|f\|^2$, where (x, f) denotes the pairing between $x \in N$ and $f \in N^*$.

Let N and M be real Banach spaces, M^* the conjugate space of M. Let ϕ be a mapping of N into M^* such that $\phi(N)$ is dense in M^* with $\|\phi(u)\|_{M^*} = \|u\|_N$, $\phi(ru) = r\phi(u)$, for all u in N, $r \geq 0$.

Browder [3] introduced ϕ-accretive mappings, generalizing the concept of a monotone mapping from N to N^* and of a accretive mapping from N to N.

In Browder [3] a map f of N into M is said to be strongly ϕ-accretive if there exists $c > 0$ such that for all u and v in N, $(f(u) - f(v), \phi(u - v)) \geq c\|u - v\|^2$.

It is also mentioned there that similar definitions may be formulated for maps of N into 2^M, and in particular for single-valued mappings f defined only on a subset $D(f)$ of N.

Let f be a multiple-valued mapping from a subset of N into M, that is, $f\colon D(f) \subseteq N \to 2^M$. The map f is said to be strongly weak-ϕ-accretive with constant c if there exists $c > 0$ such that for all u, v in N and x in $f(u)$ and y in $f(v)$,

$$(y - x, \phi(v - u)) \geq c\|v - u\|^2,$$

where ϕ is only required to satisfy $\|\phi(u)\|_{M^*} \leq \|u\|_N$ for all u in N.

$$\begin{aligned} c\|v - u\|^2 &\leq (y - x, \phi(v - u)) \leq \|y - x\| \cdot \|\phi(v - u)\| \\ &\leq \|y - x\| \cdot \|v - u\|, \end{aligned}$$

so $f(u)$ are disjoint for different u, and we can define a single-valued mapping f^{-1}, with $D(f^{-1}) = R(f)$ and $R(f^{-1}) = D(f)$, by $f^{-1}(x) = u$ if and only if $x \in f(u)$.

Lemma 2. *Let $f: D(f) \subseteq N \to M$ be a multiple-valued strongly weak-ϕ-accretive mapping with constant c. If $c \geq 1$, then*

(i) *f^{-1} is nonexpansive;*
(ii) *f^{-1} and f have the same fixed point(s).*

Proof. (i) and (ii) follow directly from the definitions of f^{-1} and strongly weak ϕ-accretive mapping with constant c.

The radius $R_p(A)$ of a bounded set $A \subseteq N$ from a point $p \in N$ is $\sup\{\|p - x\|: x \in A\}$. The diameter of A, diam A, is $\sup\{\|x - y\|: x, y \in A\}$. If C is another set in N, then the Čebyšev radius for A in C, $R(A, C)$, is $\inf\{R_c(A): c \in C\}$, and the Čebyšev centers of A in C, $C(A, C)$, is $\{c \in C: R_c(A) = R(A, C)\}$.

A convex set $A \subseteq N$ is said to have normal structure if for each closed convex bounded set W in A with more than one point there is a point p in W such that $R_p(W) <$ diam W.

Brodskii and Milman [1] introduced the notion of normal structure and proved their well-known theorem:

Theorem 1 (Brodskii and Milman). *If $K \subseteq N$, where N is complete, is a convex weakly compact set with normal structure, then there is a common fixed point for the set of all isometries of K onto K.*

A normed linear space N is said to be uniformly convex or rotund in every direction if and only if, for every nonzero member z of N and $\varepsilon > 0$, there exists a $\delta > 0$, such that $|\lambda| < \varepsilon$ if $\|x\| = \|y\| = 1$, $x - y = \lambda z$ and $\|x + y\| > 2(1 - \delta)$.

Theorem 2 (Day, James, and Swaminathan). *Let N be a normed linear space that is uniformly convex in every direction, and let H be a nonempty bounded subset of a convex subset S of N. Then $C(H, S)$ has at most one member.*

Lemma 3.ii.c. of Day [4] states that

Lemma 3. *If $E \subseteq N$ is a set that has in it exactly one Čebyšev center c, then c is a fixed point of every isometry of E onto E.*

It so happened that with an equality sign changed to an inequality sign, the proof of this lemma remains valid for onto nonexpansive maps, so we call the following lemma

Day's Lemma. *If $K \subseteq N$ is a set that has in it exactly one Čebyšev center c, then c is a common fixed point for all those mappings T of K onto K that are either nonexpansive or noncontractive*

Proof. For mappings T that are nonexpansive, please refer to the proof of Lemma 3.ii.c. of Day [4].

If T of K onto K is noncontractive, then T is one-to-one, so its inverse T^{-1} exists. Moreover T^{-1} is from K onto K and nonexpansive, so $T^{-1}c = c$ and hence $Tc = c$.

We are fortunate enough to have noticed this lemma, as it enables us to prove, among others, a generalization of the above-mentioned theorem of Brodskii and Milman when the underlying linear space is uniformly convex in every direction.

Theorem 3. *If K is a convex, weakly compact set in a normed linear space N that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center c of K) for the set of all those mappings T of K onto K that are either nonexpansive or noncontractive.*

Proof. Since $C(K, K)$ is not empty and a weakly compact set in a normed linear space is bounded (see, e.g., Day [5]), Theorem 3 follows from Theorem 2 and Day's Lemma.

Theorem 4. *If c is the center of a closed or open ball B of radius r in a normed linear space N, then c is a common fixed point for the set of all those mappings T of B onto B that are either nonexpansive or noncontractive.*

Proof. It is easy to see that c is the unique Čebyšev center of B in B, so Theorem 4 follows from Day's Lemma.

Theorem 5. *If K is a convex, weakly compact set in a normed linear space N that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center c of K) for the set P of all those multiple-valued pseudo-contractive mappings $T\colon D(T) = K \to N$, of K into N satisfying the following conditions. For each $T \in P$,*

(i) *there exists a positive number $r(T)$ such that $M_{r(T)}(u) \cap K$ is not empty for each $u \in K$, where, if we set $r(T) = r$,*

$$\begin{aligned} M_{r(T)} &= ((1 + r(T))I - r(T)T) \\ &= ((1 + r)I - rT)\colon D(T) = K \to N, \end{aligned}$$

(ii) *range of $M_{r(T)}$ contains K.*

Proof. In view of Lemma 1 and Theorem 3, it suffices to show that for each $T \in P$, the restriction of $J_r = M_{r(T)}^{-1}$ to K, J_r, maps K onto K.

Since the range of $M_{r(T)}$ contains K, J_r maps K into K. To show that J_r is onto, let $u \in K$, it follows from condition (i) that there is a $y \in M_{r(T)}u \cap K$. Clearly $J_r(y) = u$. Thus J_r maps K onto K.

Theorem 6. *Let c be a point in a normed linear space N and B_R be a closed or open ball of radius R with center c. Let P_R be the set of all those multiple-valued pseudocontractive mappings $T\colon D(T) \subseteq N \to N$, whose domain, $D(T)$, contains B_R. Suppose that for each $T \in P_R$, conditions* (i) *and* (ii) *of Theorem 5 are satisfied if K is changed to B_R and T to the restriction of T to B_R.*

If P is the union of P_R for all $R > 0$, then c is a common fixed point of the members of P.

Proof. The truth of Theorem 6 can be seen from Lemma 1, Theorem 4, and the proof of Theorem 5.

Theorem 7. *If K is a convex, weakly compact set in a normed linear space N that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center c of K) for the set P of all those multiple-valued strongly weak-ϕ-accretive mappings $f: D(f) = K \to N$ with constant $c \geq 1$ satisfying the following conditions. For each f in P,*

(i) *$f(u) \cap K$ is not empty for each u in K;*
(ii) *range of f contains K.*

Proof. In view of Lemma 2 and Theorem 3, it suffices to show that for each f in P, the restriction of f^{-1} to K, f^{-1}, maps K onto K.

Since the range of f contains K, f^{-1} maps K into K. Let u in K, it follows from condition (i) that there is a x in $f(u) \cap K$. Clearly $f^{-1}(x) = u$, so f^{-1} maps K onto K.

Theorem 8. *Let c be a point in a normed linear space N and B_R be a closed or open ball of radius R with center c. Let P_R be the set of all those multiple-valued strongly weak-ϕ-accretive mappings $f: D(f) \subseteq N \to N$ with constant $c \geq 1$, whose domain, $D(f)$, contains B_R. Suppose that for each f in P_R, conditions* (i) *and* (ii) *of Theorem 7 are satisfied if K is changed to B_R and f to the restriction of f to B_R. If P is the union of P_R for all $R > 0$, then c is a common fixed point of the members of P.*

Proof. The truth of Theorem 8 can be seen from Lemma 2, Theorem 4, and the proof of Theorem 7.

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