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ISOMORPHISM OF THE TOEPLITZ C^*-ALGEBRAS FOR THE HARDY AND BERGMAN SPACES ON CERTAIN REINHARDT DOMAINS

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ABSTRACT. I. Raeburn has conjectured that the Toeplitz C^*-algebras $\mathscr{T}(D)$ and $\mathscr{T}(\partial D)$ defined on the Bergman space $H^2(D)$ and the Hardy space $H^2(\partial D)$ of an arbitrary strongly pseudoconvex domain D in $\mathbb{C}^n$ are isomorphic. Applying the groupoid C^*-algebra approach of Curto, Muhly, and Renault to C^*-algebras of Toeplitz type, we prove that this conjecture holds for (not even necessarily pseudoconvex) Reinhardt domains in $\mathbb{C}^2$ satisfying a mild boundary condition.

INTRODUCTION

In [R] it was conjectured that the Toeplitz C^*-algebras $\mathscr{T}(D)$ and $\mathscr{T}(\partial D)$ defined on the Bergman space $H^2(D)$ and the Hardy space $H^2(\partial D)$ are isomorphic for arbitrary (strongly) pseudoconvex complex domains D. The conjecture has been known to hold for the unit ball in $\mathbb{C}^n$ by [C], for the weakly pseudoconvex domain $D = \{z \in \mathbb{C}^n | \sum_i |z_i|^{2p_i} < 1\}$ with p_i in $\mathbb{N}$ by [CrR], and for weakly pseudoconvex domains of finite type with smooth boundary by [S]. But for general (strongly) pseudoconvex domains D, it remains unsolved. Applying the groupoid C^*-algebra approach of Curto, Muhly, and Renault [CuM, MRe], to C^*-algebras of Toeplitz type, and the results of [SShU, Sh], we can prove this conjecture for D, a (not even necessarily pseudoconvex) Reinhardt domain in $\mathbb{C}^2$ satisfying a mild boundary condition (essentially the condition used in [Sh]). We shall adapt the arguments used for the case of Bergman spaces in [Sh] to the case of Hardy spaces, and we shall use the notations and technical lemmas in [Sh] as well.

1. TOEPLITZ AND GROUPOID C^*-ALGEBRAS

Let D be a bounded Reinhardt domain in $\mathbb{C}^2$, i.e., a bounded connected open region in $\mathbb{C}^2$ containing 0 that is invariant under componentwise multiplication by elements of the two-torus $\mathbb{T}^2$. The invariance under $\mathbb{T}^2$ implies that D is completely determined by $|D| := \{|z| \,|\, z \in D\} \subseteq \mathbb{R}^2_{\geq}$, where $|z| = (|z_1|, |z_2|)$ and $\mathbb{R}_{\geq} = \{x | x \in \mathbb{R},\ x \geq 0\}$. Without loss of generality,

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we may assume that D is contained in the unit polydisk Δ^2. We define the corresponding logarithmic domain $C := \{\ln(|z||z \in D, \ z_1 z_2 \neq 0\} \subseteq \mathbb{R}^2_<$ with ln acting componentwise. (It is known that a complete Reinhardt domain D is pseudoconvex if and only if C is convex [H].) In the following, we assume that D satisfies the conditions (I)–(III) in [Sh, §4], and to avoid technical difficulties in defining the Hardy space, we shall also assume that the boundary ∂C is a union of smooth curves such that the total (arclength) measure of $\partial|D|$ (and hence the total three-dimensional surface measure of ∂D) is positive and finite. (In particular, we assume that the functions ϕ and ψ defined on [Sh, pp. 276, 296] are piecewise smooth and have bounded first derivatives on K_ε for sufficiently small $\varepsilon > 0$.) So for example, the complete pseudoconvex Reinhardt domains D with piecwise smooth boundary (cf. [Sh, Remark 4.2]) and the complete Reinhardt domains D with piecewise analytic boundary ∂C such that the measure of $\partial|D|$ is finite are included.

Let $H^2(D)$ be the Bergman space over D, i.e., the Hilbert subspace of $L^2(D)$ consisting of holomorphic L^2-functions over D (with respect to volumetric Lebesgue measure), and let $H^2(\partial D)$ be the Hardy space over D, i.e., the closure of the space of continuous functions on ∂D that can be extended continuously to holomorphic functions on D, in the Hilbert space $L^2(\partial D)$ (with respect to the surface measure on ∂D). Let P and P' be the orthogonal projections from $L^2(D)$ onto $H^2(D)$ and from $L^2(\partial D)$ onto $H^2(\partial D)$, respectively. The Toeplitz C^*-algebra $\mathscr{T}(D)$ (resp. $\mathscr{T}(\partial D)$) is defined to be the C^*-algebra generated by the operators $T_\phi := PM_\phi$ restricted to $H^2(D)$ (resp. $T'_\phi := P'M'_\phi$ restricted to $H^2(\partial D)$) where ϕ is a continuous function on the closure $\overline{D}$ of D and M_ϕ (resp. M'_ϕ) is the multiplication operator by ϕ on $L^2(D)$ (resp. on $L^2(\partial D)$). In order to study $\mathscr{T}(D)$ and $\mathscr{T}(\partial D)$, we use the framework introduced by Curto and Muhly in [CuM] to relate them to groupoid C^*-algebras $C^*(\mathfrak{G})$ and $C^*(\mathfrak{G}')$, respectively, and then we show that $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic as topological groupoids (with Haar systems) and hence $\mathscr{T}(D)$ and $\mathscr{T}(\partial D)$ are isomorphic. First, let us sketch the result of [CuM].

With respect to the canonical orthonormal basis $\{e_\nu = z^\nu/\|z^\nu\|_2\}$ of $H^2(D)$ where $\nu \in \mathbb{Z}^2_\geq$ and $z^\nu = (z_1^{\nu_1})(z_2^{\nu_2})$, the Toeplitz operator T_{z_m}, $m = 1, 2$, can be viewed as a multivariable weighted shift. More precisely, $T_{z_m}(e_\nu) = w_m(\nu)e_{\nu+\varepsilon_m}$ where the ε_m's are the standard basis of $\mathbb{R}^2$ and

$$w_m(\nu) = \|z^{\nu+\varepsilon_m}\|_2/\|z^\nu\|_2.$$

Let $\mathscr{A}$ be the $\mathbb{Z}^2$-invariant commutative C^*-subalgebra of $l^\infty(\mathbb{Z}^2)$ generated by w_1 and w_2. Then $\mathscr{A} \cong C_0(Y)$, where Y is the maximal ideal space of $\mathscr{A}$ and $\mathbb{Z}^2$ is embedded in Y in the canonical way. In particular, each ν in $\mathbb{Z}^2$ determines a character $\chi(\nu)$ in Y. Clearly the $\mathbb{Z}^2$-action on Y induced by the $\mathbb{Z}^2$-action on $\mathscr{A} \subseteq l^\infty(\mathbb{Z}^2)$ coincides with the usual translation on $\mathbb{Z}^2$. The closure X of $\mathbb{Z}^2_\geq$ in Y consists of characters that are weak* limits of $\chi(\nu)$ with ν in $\mathbb{Z}^2_\geq$. Let $\mathfrak{G}$ be the reduction [MRe] of the transformation group groupoid $Y \times \mathbb{Z}^2$ to X endowed with the natural Haar system $\lambda^x = \delta_x \times \lambda$ where λ is the counting measure on $\mathbb{Z}^2$. Then the (reduced) groupoid C^*-algebra $C^*(\mathfrak{G})$ contains $\mathscr{T}(D)$ and equals $\mathscr{T}(D)$ under certain conditions.

Since D is a Reinhardt domain, it is easy to show that $\{z^\nu\}$ also form a complete orthogonal basis of $H^2(\partial D)$. In fact, it is clear that

$$\begin{aligned}\langle z^\nu, z^\mu\rangle' &= \int_{\partial|D|}\int_0^{2\pi}\int_0^{2\pi} \exp(i\theta\cdot(\nu-\mu))|z|^\nu|z|^\mu|z|^{\mathbf{1}}\,d\theta_1\,d\theta_2\,d'|z| \\ &= \int_{\partial|D|} 0\,d'|z| = 0\end{aligned}$$

if $\nu \neq \mu$ in $\mathbb{Z}^2_{\geq}$, where $\theta = (\theta_1, \theta_2)$, $\mathbf{1} = (1, 1)$, $d'|z|$ is the arclength measure on the boundary $\partial|D|$ of $|D|$ in $\mathbb{R}^2_{\geq}$, and $\langle\ ,\ \rangle'$ is the inner product on $H^2(\partial D)$. On the other hand, given f in $C(\overline{D})$ such that $f|_D$ is holomorphic, by [H] we have a power series expansion $f(z) = \sum_\nu a_\nu z^\nu$ absolutely converging on every compact subset of $\widetilde{D}$, the pseudoconvex hull of D (whose logarithmic domain is the convex hull $\widetilde{C}$ of C). Define $f_\varepsilon(z) = f((1-\varepsilon)z)$ for $\varepsilon > 0$ and $z \in \widetilde{D}$. Then clearly f_ε converges to f uniformly on ∂D and $\sum_\nu a_\nu((1-\varepsilon)z)^\nu$ converges absolutely and uniformly to $f_\varepsilon(z)$ on the closure of $\widetilde{D}$ that contains $\overline{D}$. So $f_{\varepsilon|\partial D}$ is in the closed subspace spanned by z^ν in $L^2(\partial D)$ for all $\varepsilon > 0$ and hence so is f since the total measure of ∂D is finite. Thus $H^2(\partial D)$ equals the closed span of z^ν, $\nu \in \mathbb{Z}^2_{\geq}$, in $L^2(\partial D)$.

Now we can apply the above procedure used for $H^2(D)$ to associate a groupoid $\mathfrak{G}'$ to $H^2(\partial D)$, and we get the corresponding objects w', $\mathscr{A}'$, Y', X', and $\chi'(\nu)$. First we shall try to analyze the structure of $\mathfrak{G}'$. In order to do this, we need to find all possible weak* limits of $\chi'(\nu)$ with ν a sequence in $\mathbb{Z}^2_{\geq}$.

2. Technical lemmas

In the following, for any vectors y and ν in $\mathbb{R}^2$, we shall write $y\nu := y_1\nu_1 + y_2\nu_2$.

Let u be a unit vector in $\mathbb{R}^2_{\geq}$, $u^\perp = (-u_2, u_1)$ and $\widetilde{F} = \widetilde{F}_u := (\mathbb{R}u^\perp - \gamma u) \cap \partial\widetilde{C}$ be the face of $\widetilde{C}$ determined by u, where $\widetilde{C}$ is the convex hull of C and γ is the shortest distance between ∂C and $\mathbb{R}u^\perp$. For a more detailed definition of notions used in the following we refer the reader to [Sh].

From now on, we shall use ν to denote a sequence of elements in $\mathbb{Z}^2_{>} \subseteq \mathbb{R}^2_{>}$ satisfying the following properties, unless otherwise specified. We write $\nu \underset{u}{\to}' (r', \rho', \omega') \in \overline{\mathbb{R}}^3$ if (1) $\lim(\nu u) = +\infty$, (2) $\lim(\nu u^\perp/\nu u) = 0$, (3) $((2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u)$ (and hence $((2(\nu-\mu)-\mathbf{1})u^\perp, (2(\nu-\mu)-\mathbf{1})u)$ for any fixed μ in $\mathbb{Z}^2$) belongs to slope r' [Sh], (4) $\rho(r', (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u) = \rho'$, and (5) if $r' = r_\pm$ then $\omega(r', (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, A_\pm) = \omega'$ (cf. [Sh, §4]). Note that conditions (1) and (2) imply that $\nu u^\perp/\|\nu\|$ converges to 0, $\nu/\|\nu\|$ converges to u, and $\|\nu\|$ diverges to ∞.

Clearly, we have

$$\|z^{\nu-\mathbf{1}}\|_2'^2 = \int_{\partial C}\int_0^{2\pi}\int_0^{2\pi} e^{(2\nu-\mathbf{1})x}\,d\theta_1\,d\theta_2\,d'x = (2\pi)^2\int_{\partial C} e^{(2\nu-\mathbf{1})x}\,d'x$$

where $d'x$ is the pull-back measure on ∂C of the measure $d'|z|$ on $\partial|D|$

through the componentwise exponential map, and hence

$$w'_m(\nu - \mathbf{1})^2 = \frac{\int_{\partial C} \exp((2(\nu + \varepsilon_m) - \mathbf{1})x)\, d'x}{\int_{\partial C} \exp((2\nu - \mathbf{1})x)\, d'x}$$
$$= \frac{\exp(-2\gamma\varepsilon_m u)\widetilde{L}'(\partial C, (2(\nu + \varepsilon_m) - \mathbf{1})u^\perp, (2(\nu + \varepsilon_m) - \mathbf{1})u, -xu)}{\widetilde{L}'(\partial C, (2\nu - \mathbf{1})u^\perp, (2\nu - \mathbf{1})u, -xu)}$$

for any ν in $\mathbb{Z}^2_>$, where

$$\widetilde{L}'(\mathfrak{L}, b, c, h) := \int_{\mathfrak{L}} \exp(b(xu^\perp) - ch(x) + c\gamma)\, d'x$$

for any measurable function h on ∂C and any measurable subset $\mathfrak{L}$ of ∂C.

Some of the technical results of [Sh] need be modified when dealing with Hardy spaces. The following three lemmas either modify some results of [Sh] or relate the quantities for Hardy spaces (e.g., $\widetilde{L}'$) to corresponding quantities (e.g., $\widetilde{L}$) studied in [Sh], so we may apply some results of [Sh]. Lemma 1 is related to [Sh, Lemma A.3].

Lemma 1. *Let b and c be two sequences of real numbers with c and $|c/b|$ diverging to ∞, and let f be a nonnegative measurable function on $\mathbb{R}$ with the measure of $f^{-1}([0, \varepsilon))$ positive for all $\varepsilon > 0$. Given a measurable function h on $\mathfrak{L} \subseteq \partial C$ with $h(x) \geq \varepsilon > 0$ for all x in $\mathfrak{L}$, if there are $\delta, M > 0$ such that $h(x) \geq \delta|xu^\perp|$ for all x in $\mathfrak{L}$ with $\sigma xu^\perp > M$ where σ is $\lim(\mathrm{sign}(b))$ if the limit exists (i.e., b is eventually always positive or negative) and $\sigma xu^\perp = |xu^\perp|$ otherwise, then*

$$\lim \widetilde{L}'(\mathfrak{L}, b, c, h)/\widetilde{L}(\widetilde{K}_\varepsilon, b, c, f) = 0$$

for any fixed $\varepsilon > 0$, where $K_\varepsilon = f^{-1}([0, \varepsilon])$ and $\widetilde{L}(S, b, c, f) = \int_S \exp(bt - cf(t))\, dt$ as defined on [Sh, p. 269].

Proof. In this proof, we use $|\mathfrak{L}|$ to denote the measure of subsets $\mathfrak{L}$ of ∂C. Without loss of generality, we may assume that $M > 2\varepsilon/\delta$. Clearly there is a bounded subset K of positive measure $|K|$ in $K_{\varepsilon/3}$. We have $bt - cf(t) = [(b/c)t - f(t)]c > -(\varepsilon/2)c$ for all $t \in K$, for sufficiently large c since K $(\subseteq K_{\varepsilon/3})$ is bounded and $\lim(b/c) = 0$, by assumption. On the other hand, for x in $\mathfrak{L}$ with $xu^\perp \in [-M, M]$, we have

$$bxu^\perp - ch(x) < c(bc^{-1}xu^\perp - \varepsilon) < -(2\varepsilon/3)c,$$

for a similar reason. Furthermore, for x in $\mathfrak{L}\backslash[-M, M]$ with $\sigma xu^\perp > M$, we have

$$\begin{aligned} bxu^\perp - ch(x) &< c(bc^{-1}xu^\perp - \delta|xu^\perp|) \\ &< -c\delta|xu^\perp|/2 < -c\delta M/2 < -c\varepsilon < -2c\varepsilon/3 \end{aligned}$$

for c sufficiently large, and for x in $\mathfrak{L}\backslash[-M, M]$ with $\sigma xu^\perp < M$ (this can happen only when σ is well defined), we have $\sigma xu^\perp < -M$ and hence

$$bxu^\perp - ch(x) \leq bxu^\perp - c\varepsilon < -M|b| - c\varepsilon \leq -c\varepsilon < -2c\varepsilon/3.$$

Thus

$$\widetilde{L}(K_\varepsilon, b, c, f) > \exp(-c\varepsilon/2)|K|$$

while

$$\widetilde{L}'(\partial C, b, c, h) < \int_{\partial C} \exp(-2c\varepsilon/3)\, d'x = \exp(-2c\varepsilon/3)|\mathfrak{L}|.$$

Now it is easy to see that

$$\lim \widetilde{L}'(\mathfrak{L}, b, c, h)/\widetilde{L}(K_\varepsilon, b, c, f) = 0$$

since $|\mathfrak{L}| \leq |\partial C| = \text{meaure}(\partial|D|) < \infty$ by assumption and $|K| > 0$. Q.E.D.

Recall that on [Sh, p. 300] we have $L(s, b, c, n) := \int_0^S \exp(bt - ct^n)\,dt$ for $s > 0$. Now we generalize this notion to include a weight function l by defining

$$L(s, b, c, n, l) := \int_0^s \exp(bt - ct^n) l(t)\,dt.$$

Lemma 2 relates $L(s, b, c, n, l)$ to $L(s, b, c, n)$ of [Sh]. Recall that two sequences b and c are similar, denoted by $b \approx c$, if $\lim(b/c) = 1$.

Lemma 2. *Let l be a positive continuous function defined on $[0, s]$, $0 < s < \infty$, and $n \in \mathbb{N}$. Then*

(1) *there are $\alpha, \beta > 0$ such that $\alpha \leq L(s, b, c, n, l)/L(s, b, c, n) \leq \beta$;*

(2) *if $\lim(b/c^{1/n}) = 0$, then we have*

$$L(s, b, c, n, l) \approx l(0)\lambda_n c^{-1/n}$$

where $\lambda_n = \int_0^\infty \exp(-t^n)\,dt$.

Proof. (1) is obvious, since l is positive and continuous on the compact set $[0, s]$ and we may take $\alpha = \min(l[0, s])$ and $\beta = \max(l[0, s])$.

(2) By a change of variable, we have

$$L(s, b, c, n, l) = c^{-1/n} \int_0^{sc^{1/n}} \exp((b/c^{1/n})t - t^n) l(t/c^{1/n})\,dt.$$

But since $\exp(-t^n)$ decreases much faster than $\exp((bc^{-1/n})t)$ increases as t goes to ∞ (for c sufficiently large), it is easy to check, by the bounded convergence theorem, that

$$\lim \int_0^{sc^{1/n}} \exp((b/c^{1/n})t - t^n) l(t/c^{1/n})\,dt = \int_0^\infty \exp(-t^n) l(0)\,dt$$

(note that l is continuous at 0 and uniformly bounded on $[0, s]$). Q.E.D.

Let $\widetilde{L}(S, b, c, f, l) := \int_S \exp(bt - cf(t)) l(t)\,dt$ for subsets $S \subseteq [0, s]$. Lemma 3 is related to [Sh, Lemma A.4].

Lemma 3. *Given sequences b and c with c and $|c/b|$ diverging to ∞ and a nonnegative continuous function f on $[0, s]$ with $f^{-1}(0) = \{0\}$, we have*

(1) *$\widetilde{L}([0, s_1], b, c, f, l) \approx \widetilde{L}([0, s_2], b, c, f, l)$ for any s_1, s_2 in $(0, s]$;*

(2) *if f is of degree of contact n at 0 [Sh] with $D^n f(0) = \kappa$ and $\lim(b/c^{1/n}) = 0$, then $\widetilde{L}([0, s], b, c, f, l) \approx \widetilde{L}(s, b, \kappa c, n, l)$.*

Proof. (1) Use Lemma A.3 in the appendix of [Sh] and the fact that the weight function $l(t)$ is bounded away from both 0 and ∞ for t in $[0, s]$.

(2) For each $\varepsilon > 0$ there is $\delta > 0$ such that

$$(\kappa - \varepsilon)t^n < f(t) < (\kappa + \varepsilon)t^n$$

for all t in $[0, \delta]$, and hence by Lemma 2, we get

$$l(0)\lambda_n((\kappa-\varepsilon)c)^{-1/n} \approx L(\delta, b, (\kappa-\varepsilon)c, n, l) \le \widetilde{L}([0,\delta], b, c, f, l) \le L(\delta, b, (\kappa+\varepsilon)c, n, l) \approx l(0)\lambda_n((\kappa+\varepsilon)c)^{-1/n}.$$

Now use (1) and let ε go to 0; it is easy to see that $\widetilde{L}([0,s], b, c, f, l) \approx l(0)\lambda_n(\kappa c)^{-1/n} \approx L(s, b, \kappa c, n, l)$. Q.E.D.

Note that for a function f of degree of contact n we may assume that $\widetilde{L}([0,s], b, c, f, l) \ge L(s, b, \theta c, n, l)$ for some $\theta > 0$ as in the proof of Lemma 3(2) by taking s sufficiently small as long as it is allowed to shrink s.

3. The structure of $\mathfrak{G}$

Applying Lemma 1 with $f = \phi$ on [Sh, pp. 276, 296] and $\mathfrak{L} = \partial C \backslash K'_\varepsilon$ where $K'_\varepsilon = \{x \in \partial C \mid \mathrm{dist}(x, \mathbb{R}u^\perp) \le \varepsilon + \gamma\}$, we have

$$\exp(-2\gamma u_m) \lim w'_m(\nu - \mathbf{1})^2 = \lim \frac{\widetilde{L}'(K'_\varepsilon, (2(\nu+\varepsilon_m)-\mathbf{1})u^\perp, (2(\nu+\varepsilon_m)-\mathbf{1})u, -xu)}{\widetilde{L}'(K'_\varepsilon, (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, -xu)}$$

if the latter limit exists. Since points in K'_ε are parametrized by ϕ and ψ (cf. [Sh, §4, conditions (I)–(III)], we have

$$\begin{aligned}\widetilde{L}'(K'_\varepsilon, (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, -xu) &= L(S_\phi, (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, \phi, l_\phi) \\ &\quad + L(S_\psi, (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, \psi, l_\psi)\end{aligned}$$

where $S_\phi := \{t \mid tu^\perp - (\phi(t)+\gamma)u \in K'_\varepsilon\}$ and $l_\phi(t) = \|(e^{x(t)})'\|$ with $x(t) = tu^\perp - (\phi(t)+\gamma)u$ (and similarly for ψ). (As in [Sh], we identify $\mathbb{R}u^\perp$ with $\mathbb{R}$ by identifying $y = tu^\perp$ with t.) Note that by the assumptions on ϕ and ψ in condition (III), S_ϕ and S_ψ are disjoint unions of intervals (or points) containing $F+\gamma u$ and $S_\psi \subseteq S_\phi$. However by Lemma 3, we may shrink S_ϕ and assume that $S_\phi = S_\psi$ without affecting the computation of $\lim(w_m(\nu-\mathbf{1})^2)$.

Note that since $x(t)' = u^\perp - \phi'(t)u$ is never 0 and $(e^{x(t)})' = (x'_1(t)\exp(x_1(t)), x'_2(t)\exp(x_2(t)))$, by the general assumption on ϕ, l_ϕ is uniformly bounded on K_ε and is bounded below away from 0 on any compact subset of K_ε. Furthermore, l_ϕ has positive one-sided limits at each a_i, say $\phi_{i\pm}$. A similar statement holds for ψ.

When the slope r' to which $((2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u)$ belongs (see [Sh, p. 267] for definition) is $+\infty$ (resp. $-\infty$), we get

$$\lim(w'_m(\nu-\mathbf{1})^2) = \exp(2\varepsilon_m q_k) \quad (\text{resp. } \exp(2\varepsilon_m q_1))$$

by Corollary A.2 in the appendix of [Sh] and a similar argument used in [Sh, §2] (and the remark after Lemma 3). So $\chi'(\nu-\mathbf{1})$ determines a limit character corresponding to an endpoint of F as in Case I on [Sh, pp. 284–285].

If $r' \in \mathbb{R}$ and $r' \ne 0$ when $N := \max\{n_i\} = \infty$, then by [Sh, Lemmas 2 and

3 and Corollary A.2], we have the similarity formula

$$\exp(2\gamma u_m)w'_m(\nu-\mathbf{1})^2 \approx \left\{\sum_{i,\sigma}[\widetilde{I}'(\nu+\varepsilon_m, f, i, \sigma)+\widetilde{I}'(\nu+\varepsilon_m, g, i, \sigma)]\right\}\Big/ \left\{\sum_{i,\sigma}[\widetilde{I}'(\nu, f, i, \sigma)+\widetilde{I}'(\nu, g, i, \sigma)]\right\}$$

$$\approx \left\{\sum_{i,\sigma}[\phi_{i\sigma}\widetilde{I}(\nu+\varepsilon_m, f, i, \sigma)+\psi_{i\sigma}\widetilde{I}(\nu+\varepsilon_m, g, i, \sigma)]\right\}\Big/ \left\{\sum_{i,\sigma}[\phi_{i\sigma}\widetilde{I}(\nu, f, i, \sigma)+\psi_{i\sigma}\widetilde{I}(\nu, g, i, \sigma)]\right\}$$

where

$$\widetilde{I}'(\nu, f, i, \sigma) = \exp(a_i(2\nu-\mathbf{1})u^\perp) \times \widetilde{L}([0, \sigma(I_{i\sigma}-a_i)], (2\nu-\mathbf{1})u^\perp, (2\nu-\mathbf{1})u, f_{i\sigma}, l_{\phi i\sigma})$$

with $\sigma = \pm$, $l_{\phi i\sigma}(t) = l_\phi(a_i+\sigma t)$, and $\widetilde{I}$, $f_{i\sigma}$, $g_{i\sigma}$ are as defined in [Sh, §4] (recall that $f_{i\sigma}(t) = \phi(a_i+\sigma t)$). In case of Bergman spaces, we have the same similarity formula except that the nontrivial positive coefficients $\phi_{i\sigma}$ and $\psi_{i\sigma}$ are replaced by 1 and the plus sign in front of the g terms is replaced by the minus sign (cf. [Sh, pp. 283, 298]). These differences will not matter and the procedure used in [Sh] to derive the limit characters $\lim(\chi'(\nu-\mathbf{1}))$ can be carried out here with only inessential modification. So we shall omit it.

If $r'=0$ and $N=\infty$, then by [Sh, Lemmas 2, 3 and Corollary A.2], we have

$$\exp(2\gamma u_m)\lim(w'_m(\nu-\mathbf{1})^2) = \int \exp(t(\ln(\rho')+2\varepsilon_m u^\perp))l_\phi(t)\,dt\Big/\int \exp(t\ln(\rho'))l_\phi(t)\,dt,$$

an equation similar to the corresponding one gotten for $H^2(D)$ on [Sh, p. 286] except the presence of weight L_ϕ, where the integral is over $\phi^{-1}(\{0\})$ (or $F_0+\gamma u$).

Thus we get a result similar to what was obtained in [Sh, §§2, 4]; namely, every sequence ν in $\mathbb{Z}^2_>$ with $\nu \underset{u}{\to}' (r', \rho', \omega')$ for some unit vector u in $\mathbb{R}^2_\geq$) determines a limit character $\chi'(r', \rho', \omega')$ that depends on r', ρ', ω' (and F_u) only, and so the groupoid $\mathfrak{G}'$ is parametrized by r', ρ', ω', and F_u (with some redundancy as noted on [Sh, p. 291] for the case of Bergman spaces). Now by [Sh, Lemma 1.3], $\nu \underset{u}{\to}' (r', \rho', \omega')$ if and only if $\nu \underset{u}{\to} (r', \rho'\exp(-\mathbf{1}u^\perp), \omega'\exp(-\mathbf{1}u^\perp))$. Thus the map sending (r', ρ', ω') to

$$(r, \rho, \omega) = (r', \rho'\exp(-\mathbf{1}u^\perp), \omega'\exp(-\mathbf{1}u^\perp))$$

induces a one-to-one correspondence from $\mathfrak{G}'$ to $\mathfrak{G}$, which is a topological group-oid isomorphism, and hence $C^*(\mathfrak{G}) \cong C^*(\mathfrak{G}')$. Then by the general framework of [CuM] about multivariable weighted shifts and the argument used in [Sh, proof of Theorem 3.1], we have $\mathscr{T}(D) \cong \mathscr{T}(\partial D)$, which is the goal of this paper. Note that since the structure of $\mathscr{T}(D)$ is determined in [Sh], so is the structure of $\mathscr{T}(\partial D)$ by our result.

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