

ASYMPTOTIC BEHAVIOR AND OSCILLATION OF CLASSES OF INTEGRODIFFERENTIAL EQUATIONS

A. H. NASR

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ABSTRACT. Under some conditions on the integrodifferential equations

$$\begin{aligned} \ddot{y}(t) + \int_0^t k(t-s)y(s) ds + \varphi(t) \int_0^t K(t-s)\dot{y}(s) ds \\ = f \left[t, y(t), \dot{y}(t), \int_0^t g(t, s, y(s), \dot{y}(s)) ds \right], \quad t \geq 0, \\ \ddot{y}(t) + \int_1^t k\left(\frac{t}{s}\right)y(s)\frac{1}{s} ds + \varphi(t) \int_1^t K\left(\frac{t}{s}\right)\dot{y}(s) ds \\ = f \left[t, y(t), \dot{y}(t), \int_1^t g(t, s, y(s), \dot{y}(s)) ds \right], \quad t \geq 1, \end{aligned}$$

the explicit asymptote of solutions is proved to be $y(t) = A \sin(\omega t + \delta)$ as $t \rightarrow \infty$. From this asymptote, the oscillatory behavior of the equations, the limit of the amplitudes, and the limit of the distance between consecutive zeros of the solutions are evident. Their definite values are also determined.

1. INTRODUCTION

Although the qualitative study of solutions of ordinary differential equations with and without delayed argument has received much literary attention [6, 7, 3, 2], a similar study for integrodifferential equations has not yet attracted the same. In [8] the asymptotic behavior of a second-order integrodifferential equation of the form

$$(1.1) \quad \begin{aligned} a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) \\ = f \left[t, x, \dot{x}, \int_0^t g(t, s, x(s), \dot{x}(s)) ds \right] \end{aligned}$$

is studied under some conditions on f, g . These conditions were expressed in terms of the fundamental solutions of the pure differential equation

$$(1.2) \quad a\ddot{x} + b\dot{x} + cx = 0.$$

In [1] for the very special integrodifferential equation of n th order of the form

$$\frac{d^n y}{dt^n} = b(-1)^n \int_0^t k(t-s)y(s) ds + f(t), \quad n = 1, 2, \dots,$$

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and under some suitable conditions on $k(t)$ and $f(t)$, it was proved that every bounded solution is either oscillatory or convergent to zero as $t \rightarrow \infty$ [1, p. 102]. The technique used in [1] depends on the use of the Laplace transform.

For oscillation of solutions of integrodifferential equations, we refer also to some partial results in [4, 5].

In this paper, under some conditions, we explicitly obtain the asymptotic behavior of solutions of integrodifferential equations of the form

$$(1.3) \quad \begin{aligned} \ddot{y}(t) + \int_0^t k(t-s)y(s) ds + \varphi(t) \int_0^t K(t-s)\dot{y}(s) ds \\ = f \left[t, y(t), \dot{y}(t), \int_0^t g(t, s, y(s), \dot{y}(s)) ds \right], \quad t \geq 0, \end{aligned}$$

or

$$(1.4) \quad \begin{aligned} \ddot{y}(t) + \int_1^t k\left(\frac{t}{s}\right)y(s)\frac{1}{s}ds + \varphi(t) \int_1^t K\left(\frac{t}{s}\right)\dot{y}(s) ds \\ = f \left[t, y(t), \dot{y}(t), \int_1^t g(t, s, y(s), \dot{y}(s)) ds \right], \quad t \geq 1. \end{aligned}$$

The obtained asymptotic form permits us to claim that all solutions of (1.3) and (1.4) are oscillatory, the sequence of amplitudes of every solution is convergent, and the sequence of distances between consecutive zeros has a limit as $t \rightarrow \infty$. This last limit is explicitly calculated and is unique for all solutions. Our technique here depends on the use of integral inequality similar to that treated in [9] but with relaxed conditions.

Lemma 1 [9]. *Let $a(t)$, $f(t, s)$ be nonnegative continuous functions. Assume that $a(t)$ is nondecreasing and $f(t, s)$ is nondecreasing by t for every fixed $s \leq t$. If*

$$u(t) \leq a(t) + \int_0^t f(t, s)u(s) ds$$

then

$$u(t) \leq a(t) \exp \left(\int_0^t f(t, s) ds \right).$$

The following lemma is a direct consequence of Lemma 1. We recall that it was treated in [9] under more restrictive conditions and with a different and somewhat more difficult proof. Moreover, our explicit upper bound is better.

Lemma 2. *Let $a(t)$, $f(t, s)$ be as in Lemma 1. Assume that $h(t, s)$ is nonnegative and $g(t, s)$ is nonnegative and nondecreasing by t for every fixed $s \leq t$. If*

$$(1.5) \quad u(t) \leq a(t) + \int_0^t f(t, s)u(s) ds + \int_0^t g(t, s) \int_0^s h(s, m)u(m) dm ds$$

then

$$(1.6) \quad u(t) \leq a(t) \exp \int_0^t \left[f(t, s) + g(t, s) \int_0^s h(s, m) dm \right] ds.$$

Proof. Interchanging the order of integration in the double integral in the right-hand side of (1.5),

$$u(t) \leq a(t) + \int_0^t \left[f(t, s) + \int_s^t g(t, m)h(m, s) dm \right] u(s) ds.$$

We see that the kernel within brackets satisfies the condition of Lemma 1. Thus,

$$u(t) \leq a(t) \exp \int_0^t \left[f(t, s) + \int_s^t g(t, m)h(m, s) dm \right] ds.$$

From which we get the result of the lemma.

Note. We notice that we did not assume that $h(t, s)$ is nondecreasing by t for every fixed s as done in [9].

Assume that

(i) equation (1.2) has two bounded linearly independent solutions $Z_1(t)$, $Z_2(t)$ with bounded derivatives $\dot{Z}_1(t)$, $\dot{Z}_2(t)$.

(ii) For $t \in I (= [0, \infty))$ and $z_1, z_2, z_3 \in R$,

$$|f(t, z_1, z_2, z_3)| \leq e_1(t) + r_1(t)|z_1| + r_2(t)|z_2| + r_3(t)|z_3|.$$

(iii) For $t, s \in I$ and $z_1, z_2 \in R$,

$$|g(t, s, z_1, z_2)| \leq e_2(t) + e_3(s) + k_1(t, s)|z_1| + k_2(t, s)|z_2|,$$

where $e_1, e_2, e_3, r_1, r_2, r_3, I \rightarrow R_+$, and $k_1, k_2: I^2 \rightarrow R_+$ are continuous functions.

(iv) The functions e_1, r_1, r_2, r_3 are in the class $L_1(I)$.

(v) The following integrals are bounded as $t \rightarrow \infty$:

$$\int_0^t k_1(t, s) ds, \quad \int_0^t k_2(t, s) ds, \\ \int_0^t r_3(s) \left[\int_0^s [e_2(s) + e_3(m)] dm \right] ds.$$

Lemma 3. *If conditions (i)–(v) are satisfied then every solution of (1.1) can be written in the form*

$$x(t) = A_1(t)Z_1(t) + A_2(t)Z_2(t),$$

where $\lim A_1(t), \lim A_2(t)$ exist as $t \rightarrow \infty$.

Proof. The proof can be carried out along the same lines as the proof of Theorem 1 in [8] except for the use of Lemma 2 instead of the integral inequality deduced in [9] and used in [8]. Consequently, we do not need to assume that the kernels $k_1(t, s), k_2(t, s)$ are nondecreasing by t as required in [8].

2. THE INTEGRODIFFERENTIAL EQUATIONS, ASYMPTOTIC BEHAVIOR, AND OSCILLATION

This part is the principal aim of this paper. Consider the integrodifferential equation

$$(2.1) \quad \ddot{y}(t) + \int_0^t k(t-s)y(s) ds + \varphi(t) \int_0^t K(t-s)\dot{y}(s) ds \\ = f \left[t, y, \dot{y}, \int_0^t g(t, s, y(s), \dot{y}(s)) ds \right],$$

$t \geq 0$. Assume that the functions f, g satisfy conditions (ii)–(v). Moreover, we assume that $k(t), K(t)$ and $r(t) \equiv 1 - \varphi(t)$ satisfy

(vi) $k: R_+ \rightarrow R$ is piecewise continuous and

$$0 < \int_0^\infty k(t) dt = \omega^2, \quad \int_0^\infty t|k(t)| dt < \infty,$$

$$K(t) = \int_t^\infty k(s) ds, \quad r \in L_1(R_+).$$

Theorem 1. *If conditions (ii)–(vi) are satisfied then every solution of the equation (2.1) has the form*

$$y(t) = A(t) \sin(\omega t + \delta(t)),$$

where $\lim A(t), \lim \delta(t)$ exist as $t \rightarrow \infty$.

Proof. Using integration by parts, we get

$$\int_0^t k(t-s)y(s) ds = \omega^2 y(t) - K(t)y(0) - \int_0^t K(t-s)\dot{y}(s) ds,$$

where $K(t) = \int_t^\infty k(s) ds$. Thus, equation (2.1) takes the form

$$(2.2) \quad \ddot{y}(t) + \omega^2 y(t) = y(0)K(t) + r(t) \int_0^t K(t-s)\dot{y}(s) ds + f \left[t, y(t), \dot{y}(t), \int_0^t g(t, s, y(s), \dot{y}(s)) ds \right].$$

From the conditions on $k(t)$, we have

$$\begin{aligned} \int_0^\infty |K(t)| dt &\leq \int_0^\infty \int_t^\infty |k(s)| ds dt \\ &= t \int_t^\infty |k(s)| ds \Big|_0^\infty + \int_0^\infty t|k(t)| dt \leq 2 \int_0^\infty t|k(t)| dt, \end{aligned}$$

i.e., $K \in L_1(0, \infty)$.

Moreover,

$$\int_0^t |K(t-s)| ds = \int_0^t |K(s)| ds \leq 2 \int_0^\infty t|k(t)| dt.$$

Consequently, the right-hand side of (2.2) satisfies all conditions necessary for applying Lemma 3. Therefore, every solution of (2.2) can be written in the form

$$(2.3) \quad y(t) = A_1(t) \sin \omega t + A_2(t) \cos \omega t,$$

where $\lim A_1(t), \lim A_2(t)$ exist as $t \rightarrow \infty$ and $\sin \omega t, \cos \omega t$ is a fundamental set of solutions of the equation $\ddot{y}(t) + \omega^2 y(t) = 0$. Of course, (2.3) can be written in the form

$$y(t) = A(t) \sin(\omega t + \delta(t)),$$

where $\lim A(t), \lim \delta(t)$ exist as $t \rightarrow \infty$.

Corollary. *Under the conditions of the theorem, equation (2.2) is oscillatory and the sequence of amplitudes of every solution has its own limit and the sequence of the distances between consecutive zeros of every solution is convergent to π/ω .*

Consider another type of integrodifferential equations

$$(2.4) \quad \begin{aligned} y(t) + \int_1^t k\left(\frac{t}{s}\right) y(s) \frac{1}{s} ds + \varphi(t) \int_1^t K\left(\frac{t}{s}\right) \dot{y}(s) ds \\ = f\left[t, y, \dot{y}, \int_1^t g(t, s, y(s), \dot{y}(s)) ds\right], \end{aligned}$$

$t \geq 1$. For the kernel $k(s)$, we assume that it is piecewise continuous on $[1, \infty)$ and

(vii)

$$0 < \int_1^\infty k(s) ds = \omega^2, \quad \int_1^\infty |k(t)| dt < \infty, \quad \int_1^\infty t|k(t)| dt < \infty.$$

Theorem 2. *If conditions (ii)–(v) and (vii) are satisfied then every solution of (2.4) has the form*

$$y(t) = A(t) \sin(\omega t + \delta(t)),$$

where $\lim A(t)$, $\lim \delta(t)$ exist as $t \rightarrow \infty$.

Proof. As in the case of Theorem 1, we have

$$\begin{aligned} \int_1^t k\left(\frac{t}{s}\right) y(s) \frac{1}{s} ds = \omega^2 y(t) - \frac{K(t)}{t} y(1) \\ - \int_1^t \frac{s}{t} K\left(\frac{t}{s}\right) \dot{y}(s) ds - \int_1^t \frac{1}{t} K\left(\frac{t}{s}\right) y(s) ds, \end{aligned}$$

where $K(t) = \int_t^\infty k(s) ds$. It is sufficient to prove that

$$\begin{aligned} \int_1^\infty \frac{|K(t)|}{t} dt < \infty, \\ \max_t \int_1^t \frac{s}{t} \left| K\left(\frac{t}{s}\right) \right| ds < \infty, \quad \max_t \int_1^t \frac{1}{t} K\left(\frac{t}{s}\right) ds < \infty. \end{aligned}$$

In fact each of these quantities is less than

$$\int_1^\infty |K(t)| dt \leq \int_1^\infty \int_t^\infty |k(s)| ds dt \leq 2 \int_1^\infty s|k(s)| ds.$$

Applying Lemma 3, we get the required result.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE FOR WOMEN, AIN SHAMS UNIVERSITY,
ASMA FAHMI ST., HELIOPOLIS, CAIRO, EGYPT