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SAEKI'S IMPROVEMENT OF THE VITALI-HAHN-SAKS-NIKODYM THEOREM HOLDS PRECISELY FOR BANACH SPACES HAVING COTYPE

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ABSTRACT. We prove that a Banach space X has nontrivial cotype if and only if given any σ-field Σ and any sequence $\mu_n:\Sigma \to X$ of strongly additive vector measures such that for some $\gamma \geq 1$, $\limsup_{n\to\infty} \|\mu_n(E)\| \leq \gamma \liminf_{n\to\infty} \|\mu_n(E)\| < \infty$ for each $E \in \Sigma$ then $\{\mu_n : n \in \mathbb{N}\}$ is uniformly strongly additive.

In a recent note [S] Saeki introduced the notion of a measuroid and, based on his work in measuroids, was able to substantially improve the classical Vitali-Hahn-Saks-Nikodym Theorem [DU, p. 23]—but a price must be paid. The price: the Banach space must satisfy the following "fatness" condition: for each constant $C > 0$ there exists a positive integer m such that given $x_1, \dots, x_m \in X$, $\|x_i\| \geq 1$ for each $i = 1, \dots, m$, there exists $F \subseteq \{1, \dots, m\}$ such that $\|\sum_{i\in F} x_i\| \geq C$.

The payoff: given a sequence (μ_n) of strongly additive X-valued vector measures defined on a σ-field Σ such that there exists a constant $\gamma \geq 1$ so that for each $E \in \Sigma \limsup_{n\to\infty} \|\mu_n(E)\| \leq \gamma \liminf_{n\to\infty} \|\mu_n(E)\| < \infty$ then $\{\mu_n\}$ is uniformly strongly additive.

In this note we relate Saeki's fatness condition precisely with the geometry of the Banach space. First, a couple of definitions.

Definition 1. We say a Banach space X has cotype q (≥ 2) if there is a constant $K_q > 0$ such that for each $n \geq 1$, $x_1, \dots, x_n \in X$, we have

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leq K_q \int_0^1 \left\|\sum_{k=1}^{n} r_k(t) x_k\right\| dt$$

where (r_n) denotes the Rademacher sequence on $[0, 1]$.

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Definition 2. We say a Banach space X contains the l_∞^n's uniformly if there is a constant $\lambda > 1$ such that for each $n \geq 1$ there is an n-dimensional subspace E_n of X and isomorphism $\varphi_n: l_\infty^n \to E_n$ such that $\|\varphi_n\|\,\|\varphi_n^{-1}\| < \lambda$.

Maurey and Pisier [MP] have shown that for any Banach space X, X has some cotype if and only if X does not contain the l_∞^n's uniformly.

Proposition 3. *Let X be any Banach space. The following are equivalent*:

(a) *X has cotype.*

(b) *X satisfies the fatness condition.*

(c) *If (μ_n) is a sequence of strongly additive (respectively, countably additive) X-valued vector measures on a σ-algebra Σ such that there exists $\gamma \geq 1$ such that for each $A \in \Sigma$ we have $\limsup_{n\to\infty} \|\mu_n(A)\| \leq \gamma \liminf_{n\to\infty} \|\mu_n(A)\| < \infty$, then $\{\mu_n\}$ is uniformly strongly additive (respectively, uniformly countably additive).*

(d) *X does not contain the l_∞^n's uniformly.*

Proof. (a) $\Rightarrow$ (b). Suppose X has cotype $q \geq 2$. Observe that if $x_1, \dots, x_n \in X$, $\|x_i\| \geq 1$ for each $i = 1, \dots, n$ then we have

$$n^{1/q} \leq \left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leq K_q \int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\| dt$$
$$= K_q 2^{-n} \sum_{\varepsilon_1=\pm 1, \dots, \varepsilon_n=\pm 1} \|\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\|$$

where $K_q > 0$ is the cotype q constant. Since the right-hand sum has 2^n terms, for some choice say $\varepsilon_1' = \pm 1, \dots, \varepsilon_n' = \pm 1$ we have

$$(*) \qquad n^{1/q} K_q^{-1} \leq \|\varepsilon_1' x_1 + \varepsilon_2' x_2 + \cdots + \varepsilon_n' x_n\|.$$

Let $P = \{i | \varepsilon_i' = 1\}$ and $N = \{i | \varepsilon_i' = -1\}$. From the triangle inequality and $(*)$ we deduce $\|\sum_{i\in P} x_i\| \geq 2^{-1} n^{1/q} K_q^{-1}$ or $\|\sum_{i\in N} x_i\| \geq 2^{-1} n^{1/q} K_q^{-1}$.

Hence, given $C > 0$, choose n so that $2^{-1} n^{1/q} K_q^{-1} \geq C$ in order to fulfill the fatness condition.

(b) $\Rightarrow$ (c). [S, Corollary 8].

(c) $\Rightarrow$ (d). Suppose X contains the l_∞^n's uniformly. Then, we have a constant $\lambda > 1$ such that for each $n \geq 1$ there is an n-dimensional subspace E_n of X and an isomorphism $\varphi_n: l_\infty^n \to E_n$ such that $\|\varphi_n\| = 1$ and $\|\varphi_n^{-1}\| < \lambda$. Therefore, for each $n \geq 1$ and each $F \subseteq \{1, \dots, n\}$, $F \neq \varnothing$,

$$(**) \qquad \left\|\sum_{i\in F} \varphi_n(e_i^{(n)})\right\| \leq 1$$

and

$$(***) \qquad \left\|\sum_{i\in F} \varphi_n(e_i^{(n)})\right\| \geq \lambda^{-1}$$

where $e_1^{(n)}, \dots, e_n^{(n)}$ denotes the unit vector basis elements of l_∞^n.

Now, for each $n \geq 1$, define $\mu_n: P(\mathbb{N}) \to X$ by $\mu_n(\Delta) = \sum_{i\in\Delta\cap\{1,\dots,n\}} \varphi_n(e_i^{(n)})$. Clearly, each μ_n is finitely additive. In fact, each μ_n is countably additive since given $(B_i) \subseteq P(\mathbb{N})$, $B_i \cap B_j = \varnothing$ for each $i \neq j$, $B_i \cap \{1, \dots, n\} = \varnothing$ for all i sufficiently large. From $(**)$ and $(***)$ it follows that for each $\Delta \in P(\mathbb{N})$,

$\limsup_{n\to\infty} \|\mu_n(\Delta)\| \le \lambda \liminf_{n\to\infty} \|\mu_n(\Delta)\| < \infty$. However, (μ_n) is not even uniformly strongly additive since $\|\mu_n(\{n\})\| \ge \lambda^{-1}$ for each $n \ge 1$.

(d) $\Rightarrow$ (a). One direction of the theorem of Maurey and Pisier already noted.

Remark 4. It is not difficult to see that in Proposition 3 we can add the following equivalent statement:

(c′) *If* (μ_n) *is a sequence of* X*-valued vector measures on a* σ*-field* Σ *and* m *is a countably additive nonnegative measure such that for each* n, *then* μ_n *is* m*-continuous, and if there exists a constant* $\gamma \ge 1$ *such that for each* $A \in \Sigma$ *we have* $\limsup_{n\to\infty} \|\mu_n(A)\| \le \gamma \liminf_{n\to\infty} \|\mu_n(A)\| < \infty$ *then* $\{\mu_n\}$ *is uniformly* m*-continuous.*

Hence, we also have an improvement of the classical Vitali-Hahn-Saks Theorem [DU, p. 24] that holds precisely when the Banach space has cotype.

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