DEFEAT OF THE $FP^2F$ CONJECTURE: HUCKABA'S EXAMPLE

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Abstract. A commutative ring $R$ is $FP^2F$ (resp. $FPF$) provided that all finitely presented (resp. finitely generated) faithful modules generate the category mod-$R$ of all $R$-modules. A conjecture of the author dating to the middle 1970s states that any $FP^2F$ ring $R$ has FP-injective classical quotient ring $Q = Q_{cl}(R)$. It was shown by the author (Injective quotient rings. II, Lecture Notes in Pure and Appl. Math., vol. 72, Dekker, New York, 1982, pp. 71-105) that $FPF$ rings $R$ have injective $Q$ and by the author and P. Pillay (Classification of commutative $FPF$ rings, Notas Math., vol. 4, Univ. de Murcia, Murcia, Spain, 1990) that $CP^2F$ local rings (defined below) have FP-injective $Q$.

The counterexample is a difficult example due to Huckaba of a strongly Prüfer ring without "Property A." (A ring with Property A was labelled a McCoy ring by the author.)

This counterexample is $CP^2F$ in the sense that every factor ring of $R$ is $FP^2F$.

1. Theorem. (1) A ring $R$ is $CP^2F$ iff (2) $R_M$ is a valuation ring ($= VR$) for each maximal ideal $M$.

Proof. In [Fl] (also [FP]) $CP^2F$ is characterized by the statement: $R$ is locally a VR, i.e., $R_P$ is a VR for all prime ideals $P$. However, a look at the proof in [Fl, Corollary 5E, p. 176] establishes the equivalence of (1) and (2).

A ring $R$ is said to be McCoy (see [F2]) or have Property A (see [H]) provided that every finitely generated dense (= faithful) ideal is regular, i.e., contains a regular element. This is equivalent to the statement that $Q = Q_{cl}(R)$ is McCoy, i.e., that $Q$ is the only finitely generated dense ideal. A sufficient condition for $R$ to be McCoy is for $Q$ to be FP-injective.

Every FP-injective ring $Q$ has the property that finitely generated ideals are annihilator ideals.

A ring $R$ is Prüfer if every finitely generated regular ideal is invertible (see [H, p. 29]), equivalently projective. A ring $R$ is strongly Prüfer iff every finitely generated dense ideal is locally principal [H, p. 115].

Huckaba's example. There exists a strongly Prüfer ring $R$ with the properties that $R$ is not McCoy; hence $Q = Q_{cl}(R)$ is not FP-injective, but $R_M$ is a VR, in fact a domain ($= VD$), for all maximal ideals. (See [H, Example 17, p. 191].)
This provides the counterexample to the $FP^2 F$ conjecture since $R$ is $CFP^2 F$ by Theorem 1, yet $Q$ is not FP-injective.

We also remark the following:

**Theorem** (Huckaba and Keller). A reduced coherent ring $R$ is McCoy iff $Q$ is VNR ($= \text{von Neumann regular}$).

**Proof.** See [H, Theorem 4.7, p. 20].

**REFERENCES**


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