PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 1, September 1992

REMARKS ON A MULTIPLIER CONJECTURE FOR UNIVALENT FUNCTIONS

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. In this paper we study some aspects of a conjecture on the convolution of univalent functions in the unit disk $\mathbb{D}$, which was recently proposed by Grünberg, Rønning, and Ruscheweyh (Trans. Amer. Math. Soc. **322** (1990), 377–393) and is as follows: let $\mathscr{D} := \{f \text{ analytic in } \mathbb{D}\colon |f''(z)| \leq \operatorname{Re} f'(z),\ z \in \mathbb{D}\}$ and $g, h \in \mathscr{S}$ (the class of normalized univalent functions in $\mathbb{D}$). Then $\operatorname{Re}(f*g*h)(z)/z > 0$ in $\mathbb{D}$. We discuss several special cases, which lead to interesting, more specific statements about functions in $\mathscr{S}$, determine certain extreme points of $\mathscr{D}$, and note that the former conjectures of Bieberbach and Sheil-Small are contained in this one. It is an interesting matter of fact that the functions in $\mathscr{D}$, which are "responsible" for the Bieberbach coefficient estimates are not extreme points in $\mathscr{D}$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\mathscr{A}$ denote the set of analytic functions f in the unit disk $\mathbb{D}$. If $f(0) = 0$ or $f(0) = 1$ then $f \in \mathscr{A}_0$ or $f \in \mathscr{A}_1$, respectively. $\mathscr{S}$ is the set of univalent functions in $\mathscr{A}$, normalised by $f \in \mathscr{A}_0$, $f' \in \mathscr{A}_1$. We say that $h \in \mathscr{S}^2$ if

$$h(z) = \int_0^z (f * g)(t)\frac{dt}{t}, \qquad z \in \mathbb{D},\ f, g \in \mathscr{S}, \tag{1}$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions in $\mathbb{D}$. Finally, let

$$\mathscr{D} := \{F \in \mathscr{A}_0\colon F' \in \mathscr{A}_1 \wedge |F''(z)| \leq \operatorname{Re} F'(z),\ z \in \mathbb{D}\},$$
$$\mathscr{D}' := \{F'\colon F \in \mathscr{D}\}.$$

Some evidence has been obtained in [2] for

Conjecture A. *For $G \in \mathscr{D}'$ and $h \in \mathscr{S}^2$ we have*

$$\operatorname{Re}\{G(z) * h(z)/z\} > 0, \qquad z \in \mathbb{D}. \tag{2}$$

Received by the editors October 26, 1990.
1991 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C55.
This work was carried out while the first author visited Würzburg University; he was supported by an ACC (Quebec) grant.
The second author's research was partially supported by the Fondo Nacional de Desarrollo Cientifico y Tecnológico, Chile (FONDECYT, Grant 478/90).

Actually, the original conjecture (compare also [6]) was

$$\forall F \in \mathscr{D}, \ f, g \in \mathscr{S}: \operatorname{Re}(F * f * g)(z)/z > 0, \qquad z \in \mathbb{D}, \tag{3}$$

which obviously is equivalent to Conjecture A. It is easily verified that

$$\mathscr{Z} := \left\{ G(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \in \mathscr{A}_1 : \sum_{k=1}^{\infty} (k+1)|a_k| \leq 1 \right\} \subset \mathscr{D}' \tag{4}$$

and that, as a consequence of the Bieberbach estimates (de Branges's Theorem [1]), Conjecture A is true for $\mathscr{Z}$. On the other hand, the choices

$$G_n(z) := 1 + z^{n-1}/n \in \mathscr{Z} \subset \mathscr{D}', \qquad n = 2, 3, \dots, \tag{5}$$

can be used to show that Conjecture A *contains* the former Bieberbach conjecture. As another example it has been shown in [2] that Conjecture A holds for the following members of $\mathscr{D}'$ as well:

$$\sum_{k=1}^{n} k(z/4)^{k-1}, \qquad n \in \mathbb{N},$$

which implies a vast generalisation of Szegö's well-known $\frac{1}{4}$-Theorem on the partial sums of functions in $\mathscr{S}$. Furthermore, Conjecture A is true if we replace $\mathscr{S}^2$ by the corresponding class $\mathscr{K}^2$ where the functions in $\mathscr{S}$ are replaced by close-to-convex functions and in $\mathscr{T}^2$ where we admit only typically real functions from $\mathscr{S}$ in (1).

Choosing $f(z) = z/(1-z)^2$ in (1) we arrive at a somewhat weaker conjecture that, however, would still contain the Bieberbach coefficient estimates.

Conjecture B. *For $G \in \mathscr{D}'$ and $h \in \mathscr{S}$ we have*

$$\operatorname{Re}\{G(z) * h(z)/z\} > 0, \qquad z \in \mathbb{D}. \tag{6}$$

Even this more modest claim seems to be hard to verify in general.

The set $\mathscr{D}'$ is convex, compact (in the topology of locally uniform convergence in $\mathbb{D}$), and rotationally invariant, i.e.,

$$G \in \mathscr{D}' \Rightarrow \forall |x| \leq 1 : G_x \in \mathscr{D}',$$

where $G_x(z) := G(xz)$, $z \in \mathbb{D}$. It is therefore of interest to study the extreme points of $\mathscr{D}'$, in particular in view of the fact that for a proof of Conjectures A, B it would be sufficient to prove them on the extreme points of $\mathscr{D}'$. The results in this paper deal with this question. We shall prove

Theorem 1. *Let $G \in \mathscr{D}'$ be analytic in $\overline{\mathbb{D}}$. Then G is an extreme point of $\mathscr{D}'$ if and only if*

$$|G'(z)| = \operatorname{Re} G(z), \qquad z \in \partial\mathbb{D}. \tag{7}$$

The following result is an immediate consequence of Theorem 1.

Theorem 2. (a) *For $n \in \mathbb{N}$ let*

$$d_n := \sqrt{n^2+1} - n. \tag{8}$$

Then the functions

$$E_n(z) := \frac{1 + d_n z^n}{1 - d_n z^n}, \qquad z \in \mathbb{D}, \tag{9}$$

along with their rotations $E_{n,x}$, $|x| = 1$, *are extreme points of* $\mathscr{D}'$.

(b) *The functions* G_n *from* (5) *are not extreme points of* $\mathscr{D}'$.

Part (b) of Theorem 2 is particularly surprising in view of the extremal role the Bieberbach coefficient estimates seem to play in the theory of $\mathscr{S}$. The functions E_n do have some interesting features in both the theory of the class $\mathscr{D}'$ and Conjecture A.

Theorem 3. *Let* $G \in \mathscr{D}'$, $G(z) = \sum_{k=0}^{\infty} a_k z^k$. *Then for* $n \in \mathbb{N}$ *we have*

$$|a_n| \le 2d_n, \tag{10}$$

with equality only for $G = E_n$ *and its rotations* G_x, $|x| = 1$.

We strongly believe that the functions $E_{n,x}$ are the only extreme points of $\mathscr{D}'$ that are analytic in $\overline{\mathbb{D}}$ and, in particular, the only rational ones. A proof of this, however, is not yet available. On the other hand, we also cannot prove that $\mathscr{D}'$ has any other extreme point besides the $E_{n,x}$, although this is very likely.

Conjecture A, respectively B, can be formulated more explicitly in terms of functions in $\mathscr{S}^2$ and $\mathscr{S}$, when we choose $G = E_n$.

Conjecture C. *Let* $n \in \mathbb{N}$, d_n *be as in* (8), *and* $h \in \mathscr{S}^2$ (*or, weaker,* $h \in \mathscr{S}$). *Then*

$$\operatorname{Re}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{h(\varepsilon^k z)}{\varepsilon^k z}\right\} > \frac{1}{2}, \qquad |z| < d_n^{1/n}, \tag{11}$$

with $\varepsilon := e^{2\pi i/n}$.

Clearly, Conjecture C is true for $h \in \mathscr{K}^2$ and $h \in \mathscr{T}^2$, and it is sharp for the Koebe functions and suitable z with $|z| = d_n^{1/n}$. For the whole of $\mathscr{S}^2$ we can verify Conjecture C only for the case $n = 1$, $h \in \mathscr{S}$, where it is a consequence of Grunsky's result [3] on the domain of $h(z)/z$, $z \in \mathbb{D}$, fixed; see also [8]. However, we have the following partial results.

Theorem 4. *For* $h \in \mathscr{S}^2$, $n \in \mathbb{N}$, *define*

$$r_n(h) := \sup\left\{0 < r < 1 : \operatorname{Re}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{h(\varepsilon^k z)}{\varepsilon^k z}\right\} > \frac{1}{2},\ |z| < r^{1/n}\right\} \tag{12}$$

and

$$r_n := \inf_{h \in \mathscr{S}^2} r_n(h). \tag{13}$$

Then

$$\liminf_{n\to\infty} r_n/d_n = 1. \tag{14}$$

Theorem 5. *Let* $h \in \mathscr{S}^2$. *Then there is a number* $n_0(h)$ *so that* (11) *holds for* h *and all* $n \ge n_0$.

We are using the truth of Bieberbach's conjecture for the proof of Theorem 4. It can be seen that (14) is, in fact, equivalent to an asymptotic version of the Bieberbach conjecture; namely,

$$\limsup_{n\to\infty} \max_{f\in\mathscr{S}} |f^{(n)}(0)/nn!| = 1.$$

This indicates, at least, that Conjecture C is not a very elementary one.

The proofs of Theorems 1–5 are given in §2. Section 3 is devoted to the discussion of some related conjectures and results.

2. Proofs of Theorems 1–5

We start with a simple but crucial lemma.

Lemma 1. *Let* $n \in \mathbb{N}$ *and* $G(z) = 1 + Az^n + \cdots \in \mathscr{D}'$. *Then*

$$G \prec \frac{1 + d_n z}{1 - d_n z}. \tag{15}$$

In particular,

$$|A| \leq 2d_n\,.] \tag{16}$$

Proof. We write $G(z) = (1 + d_n w(z))/(1 - d_n w(z))$ where $w \in \mathscr{A}_0$. We wish to show that $\|w\| \leq 1$, and we make use of the so-called Jack's Lemma [5]. Assume that $|w(z)| \leq |w(z_0)| = 1$ for some $z_0 \in \mathbb{D}$ and all $|z| \leq |z_0|$. Then $\kappa := z_0 w'(z_0)/w(z_0) \geq n$, and hence

$$\operatorname{Re} G(z_0) - |G'(z_0)| = \frac{1 - d_n^2 - 2\kappa d_n/|z_0|}{|1 - d_n w(z_0)|^2} < 0,$$

which contradicts $G \in \mathscr{D}'$. This proves the subordination (15) and readily implies (16) as well. □

We note that Lemma 1 also implies

$$\sqrt{2} - 1 \leq \operatorname{Re} G(z) \leq \frac{1}{\sqrt{2} - 1}, \qquad z \in \mathbb{D}, \tag{17}$$

for $G \in \mathscr{D}'$.

For the discussion of the extreme points of $\mathscr{D}'$ we make use of the following well-known lemma.

Lemma 2. *Let* X *be a locally convex topological vector space over* $\mathbb{R}$, *and let* $E \subset X$ *be closed and convex. Then* $m \in E$ *is an extreme point of* E *if and only if there is no* $x \in X$, $x \neq 0$, *with* $m + x \in E$ *and* $m - x \in E$.

Proof of Theorem 1. For $G \in \mathscr{D}'$ analytic in $\overline{\mathbb{D}}$ we define

$$\gamma(z) := \operatorname{Re} G(z) - |G'(z)|, \qquad z \in \partial\mathbb{D}.$$

Assume that for some $w \in \mathscr{A}_0$, $0 \not\equiv w$, w analytic in $\overline{\mathbb{D}}$, we have

$$|w'(z)| + |w(z)| \leq \gamma(z), \qquad z \in \partial\mathbb{D}. \tag{18}$$

Then for $z \in \mathbb{D}$,

$$\begin{aligned}|G'(z) \pm w'(z)| - \operatorname{Re}(G(z) \pm w(z)) &\leq |G'(z)| + |w'(z)| + |w(z)| - \operatorname{Re} G(z)\\ &\leq \max_{z \in \partial\mathbb{D}}(|w'(z)| + |w(z)| - \gamma(z)) \leq 0,\end{aligned}$$

since $|G'| + |w'| + |w| - \operatorname{Re} G$ is subharmonic in $\overline{\mathbb{D}}$. An application of Lemma 2 shows now that G is not an extreme point of $\mathscr{D}'$. We now distinguish the following three cases:

(i) $\gamma(z) > 0$, $z \in \partial\mathbb{D}$;
(ii) $\gamma(z) = 0$ for at most finitely many points $z \in \partial\mathbb{D}$;
(iii) $\gamma(z) = 0$ for infinitely many $z \in \partial\mathbb{D}$.

(i) Since γ is continuous on the compact set $\partial\mathbb{D}$, we must have $\gamma(z) \geq \gamma(z_0) > 0$, $z \in \partial\mathbb{D}$. Then $w(z) := \gamma(z_0)z/2$ satisfies (18) and G is no extreme point.

(ii) Assume that $z_1, \dots, z_m$ are the zeros of γ on $\partial\mathbb{D}$. The function

$$H(z) := \tfrac{1}{4}(G(z) + \overline{G(1/\overline{z})})^2 - G'(z)\overline{G'(1/\overline{z})}$$

is analytic in an annulus containing $\partial\mathbb{D}$, and on $\partial\mathbb{D}$ we have $H(z) = (\operatorname{Re} G(z))^2 - |G'(z)|^2$. Hence $H(z_k) = 0$, $k = 1, \dots, m$, but $H(z) \neq 0$ in punctured neighborhoods of those points. This implies that locally

$$H(z) = A_k(z - z_k)^{n_k} + o(|z - z_k|^{n_k}), \qquad n_k \geq 1, \ k = 1, \dots, m.$$

On $\partial\mathbb{D}$ we have $H(z) = |H(z)|$, and hence

$$(\operatorname{Re} G(z))^2 - |G'(z)|^2 = |A_k||z - z_k|^{n_k}(1 + o(1)).$$

From (17) we conclude that

$$\operatorname{Re} G(z) + |G'(z)| \leq 2/(\sqrt{2} - 1), \qquad z \in \partial\mathbb{D},$$

and, therefore, we can find open neighborhoods $U(z_k) \subset \partial\mathbb{D}$, $k = 1, \dots, m$, in which

$$\gamma(z) = \operatorname{Re} G(z) - |G'(z)| \geq (\sqrt{2} - 1)|A_k||z - z_k|^{n_k}.$$

Now set

$$w(z) := \varepsilon z \prod_{k=1}^{m} (z - z_k)^{n_k + 1}.$$

A simple estimate shows that

$$|w'(z)| + |w(z)| \leq \gamma(z), \qquad z \in \bigcup_{k=1}^{m} U(z_k),$$

if we choose $|\varepsilon|$ small enough. In the compact set $\partial\mathbb{D} \setminus \bigcup_{k=1}^{m} U(z_k)$ we have $\gamma(z) \geq \rho > 0$ for some number ρ. Clearly, by further decreasing $|\varepsilon|$, we can make w satisfy (18). Hence G is not an extreme point.

(iii) If $\gamma(z)$ has infinitely many zeros on $\partial\mathbb{D}$ then the function $H(z)$ defined above is obviously identically zero on $\partial\mathbb{D}$, and hence $\gamma \equiv 0$.

This completes the proof of the "only if" part of Theorem 1. Next assume that G satisfies $\gamma \equiv 0$ but is no extreme point. Then there is some function $w \in \mathscr{A}_0$, $w \not\equiv 0$, such that $G \pm w \in \mathscr{D}'$. In particular, by (17) we conclude

$$\begin{aligned} |w'(z)| &\leq |G'(z) + w'(z)| + |G'(z)| \\ &\leq \operatorname{Re}(G(z) + w(z)) + \operatorname{Re} G(z) \leq 2/\sqrt{2} - 1, \qquad z \in \mathbb{D}. \end{aligned}$$

Thus the boundary function $w'(e^{i\phi})$ taken from radial limits exists in L^1. We have in $\mathbb{D}$

$$\begin{aligned} |G'(z)| &\leq \tfrac{1}{2}|G'(z) + w'(z)| + \tfrac{1}{2}|G'(z) - w'(z)| \\ &\leq \tfrac{1}{2}\operatorname{Re}(G(z) + w(z)) + \tfrac{1}{2}\operatorname{Re}(G(z) - w(z)) = \operatorname{Re} G(z). \end{aligned}$$

Using our assumption about G, after taking radial limits, we obtain

$$|G'(z)| = \tfrac{1}{2}|G'(z) + w'(z)| + \tfrac{1}{2}|G'(z) - w'(z)|, \tag{19}$$

almost everywhere on $\partial\mathbb{D}$. We also note that $G' \neq 0$ on $\partial\mathbb{D}$. Hence (19) implies

$$\operatorname{Im} w'(z)/G'(z) = 0, \quad \text{a.e. on } \partial\mathbb{D}. \tag{20}$$

Let Q be a rational function with all its poles in $\mathbb{D}$ such that $w'(z)/G'(z) - Q(z)$ is analytic in $\mathbb{D}$. Then

$$H(z) := w'(z)/G'(z) - Q(z) - \overline{Q(1/\overline{z})}$$

is holomorphic in $\mathbb{D}$ as well. $\operatorname{Im} H(z)$ is a bounded harmonic function in $\mathbb{D}$ with vanishing boundary values a.e. on $\partial\mathbb{D}$, and thus $H(z) \equiv \text{const}$. In fact, by a proper choice of Q, we can assume that $H \equiv 0$. This shows that there is a rational function $R(z)$ that assumes only real values on $\partial\mathbb{D}$ with

$$w'(z) = R(z)G'(z). \tag{21}$$

Inserting this into the original conditions for $G \pm w$ we get

$$|G'(z)||1 \pm R(z)| \leq \operatorname{Re}(G(z) \pm w(z)),$$

and with $\gamma \equiv 0$ this takes the following equivalent forms on $\partial\mathbb{D}$:

$$|1 \pm R(z)| \leq 1 \pm \frac{\operatorname{Re} w(z)}{\operatorname{Re} G(z)}, \qquad 1 \pm R(z) \leq 1 \pm \frac{\operatorname{Re} w(z)}{\operatorname{Re} G(z)},$$

$$R(z) = \frac{\operatorname{Re} w(z)}{\operatorname{Re} G(z)}, \qquad \operatorname{Re}(w(z) - R(z)G(z)) = 0.$$

Next we choose a rational function $P(z)$, with poles in $\mathbb{D}$ only, such that $w(z) - R(z)G(z) - P(z)$ is analytic in $\mathbb{D}$. By a similar reasoning as above we can conclude that there is a rational function $S(z)$ (given by $P(z) - \overline{P(1/\overline{z})}$), which takes only imaginary values on $\partial\mathbb{D}$, such that

$$w(z) - R(z)G(z) = S(z). \tag{22}$$

Differentiating (22) and inserting (21) into it leads to

$$G(z) = -S'(z)/R'(z). \tag{23}$$

However, our assumptions imply that $zR'(z)$ is purely imaginary and $zS'(z)$ is real on $\partial\mathbb{D}$. Therefore G has to be purely imaginary on the same set, which clearly contradicts (17). □

Proof of Theorem 3. Let $G \in \mathscr{D}'$ be a function that maximizes the nth coefficient, i.e.,

$$\forall F \in \mathscr{D}' \colon |F^{(n)}(0)| \leq |G^{(n)}(0)|.$$

We may even assume, that $G^{(n)}(0) := n!A > 0$. $\mathscr{D}'$ is a convex set and, therefore,

$$H(z) := \frac{1}{n}\sum_{k=1}^{n} G(\varepsilon^k z) \in \mathscr{D}', \qquad \varepsilon = e^{2i\pi/n}.$$

Furthermore, $H(z) = 1 + Az^n + \cdots$. Hence, by Lemma 1, we deduce (10), and it follows from the principle of subordination that $H(z) = E_n(z)$. But $H(z)$ is a convex combination of the functions $G(\varepsilon^k z) \in \mathscr{D}'$ and E_n is an extreme point in $\mathscr{D}'$. This implies that $G = E_n$. □

Proof of Theorem 4. We define

$$s_n := \frac{1}{3}(2 + n - \sqrt{1 + 4n + n^2}) = \frac{1}{2n} + o\left(\frac{1}{n}\right), \qquad n \to \infty.$$

A simple calculation shows that

$$\frac{1}{1 - s_n z^n} = \sum_{k=1}^{\infty} \lambda_k \left(1 + \frac{z^{nk}}{2(nk + 1)}\right), \tag{24}$$

where $\lambda_k := 2s_n^k(nk + 1)$ and $\sum_{k=1}^{\infty} \lambda_k = 1$. Now let

$$h(z) := \sum_{k=1}^{\infty} h_k z^k \in \mathcal{S}^2, \tag{25}$$

which implies $|h_k| \leq k$, $k \in \mathbb{N}$. For each k, n we now have

$$\operatorname{Re}\left(1 + \frac{z^{nk}}{2(nk + 1)}\right) * \frac{h(z)}{z} \geq \frac{1}{2}, \qquad z \in \mathbb{D}.$$

Using this in (24) we arrive at

$$\frac{1}{n}\operatorname{Re} \sum_{k=1}^{n} \frac{h(\varepsilon^k s_n^{1/n} z)}{\varepsilon^k s_n^{1/n} z} = \operatorname{Re}\left(\frac{1}{1 - s_n z^n} * \frac{h(z)}{z}\right) > \frac{1}{2}, \qquad z \in \mathbb{D}.$$

This implies $s_n \leq r_n(h)$, and thus $s_n \leq r_n$. Since

$$r_n \leq d_n = \sqrt{n^2 + 1} - n = \frac{1}{2n} + o\left(\frac{1}{n}\right),$$

we see that $1 = \lim_{n\to\infty} s_n/d_n \leq \lim_{n\to\infty} r_n/d_n \leq 1$, which completes the proof. □

Proof of Theorem 5. Our technique will be very similar as in the previous proof, with Hayman's celebrated regularity theorem [4] as an additional ingredient. From his result we deduce for $h \in \mathcal{S}^2$ as in (25): either h is (a rotation of) the Koebe function $z(1 - z)^{-2}$ and the theorem is true, or

$$\alpha := \limsup_{k\to\infty} |h_k/k| < 1.$$

Let $\beta := (1 + \alpha)/2$. Then there exists $n_1(h) \in \mathbb{N}$ such that

$$|h_k| \leq \beta k, \qquad k \geq n_1(h). \tag{26}$$

We define

$$\begin{aligned} s_n(\beta) &:= \frac{1 + (1 + \beta)n - \sqrt{\beta}\sqrt{2n + \beta(1 + 2n + n^2)}}{1 + 2\beta} \\ &= \frac{1}{2\beta n} + o\left(\frac{1}{n}\right), \qquad n \to \infty. \end{aligned}$$

Instead of (24) we use now

$$\frac{1}{1 - s_n(\beta) z^n} = \sum_{k=1}^{\infty} \lambda_k \left(1 + \frac{z^{nk}}{2\beta(nk + 1)}\right),$$

with

$$\lambda_k := 2\beta s_n(\beta)^k(nk+1) \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k = 1 .$$

Using (26) in a similar fashion as above, we readily deduce that $r_n(h) \geq s_n(\beta)$, $n \geq n_1(h)$. Since

$$\lim_{n\to\infty} s_n(\beta)/d_n = 1/\beta > 1 ,$$

we can now find $n_0(h) \geq n_1(h)$ so that $r_n(h) \geq d_n$, $n \geq n_0(h)$. □

3. Some related conjectures

First we shall show that Conjecture A (and Conjecture B, likewise) is equivalent to the following apparently stronger statement.

Conjecture A′. *Let* $F \in \mathscr{A}$ *be arbitrary, and define*

$$\mu(F) := \inf_{z\in\mathbb{D}}\{\operatorname{Re} F(z) - |F'(z)|\} . \tag{27}$$

Then for $h \in \mathscr{S}^2$ *(or, weaker,* $h \in \mathscr{S}$*), we have*

$$\operatorname{Re}\{F(z) * h(z)/z\} \geq \mu(F) , \qquad z \in \mathbb{D} . \tag{28}$$

That Conjecture A′ implies Conjecture A is obvious. Now assume that Conjecture A holds and let $F \in \mathscr{A}$ be nonconstant. Then $\operatorname{Re} F(0) > \mu := \mu(F)$, and we define

$$G(z) := \frac{F(z)-\mu}{\operatorname{Re} F(0)-\mu} - i\frac{\operatorname{Im} F(0)}{\operatorname{Re} F(0)-\mu} , \qquad z \in \mathbb{D} .$$

Then $G \in \mathscr{A}_1$, and we have

$$\operatorname{Re} G(z) - |G'(z)| = \frac{\operatorname{Re} F(z) - |F'(z)| - \mu}{\operatorname{Re} F(0) - \mu} \geq 0 , \qquad z \in \mathbb{D} ,$$

so that $G \in \mathscr{D}'$. This implies for $h \in \mathscr{S}^2$,

$$\begin{aligned}
&\frac{1}{\operatorname{Re} F(0)-\mu}\left[\operatorname{Re}\left(F(z) * \frac{h(z)}{z}\right) - \mu\right] \\
&\quad = \frac{1}{\operatorname{Re} F(0)-\mu}\operatorname{Re}\left[(F(z)-\mu) * \frac{h(z)}{z}\right] \\
&\quad = \operatorname{Re}\left\{\left[\frac{F(z)-\mu}{\operatorname{Re} F(0)-\mu} - i\frac{\operatorname{Im} F(0)}{\operatorname{Re} F(0)-\mu}\right] * \frac{h(z)}{z}\right\} \\
&\quad = \operatorname{Re}\left[G(z) * \frac{h(z)}{z}\right] \geq 0 ,
\end{aligned}$$

and thus (28).

Concerning sharpness in Conjecture A′ we feel that it is either sharp for h the Koebe function or not sharp at all. Applying (27) to εF, $|\varepsilon| = 1$ we arrive at a weaker form of our conjecture. By $\|\cdot\|$ we denote the sup-norm in $\mathbb{D}$.

Conjecture D. *Let* $F \in \mathscr{A}$ *be arbitrary, and let* $h \in \mathscr{S}^2$ *(or* $h \in \mathscr{S}$*). Then*

$$\|F(z) * h(z)/z\| \leq \|F\| + \|F'\| . \tag{29}$$

Note that (29) is true if F has nonnegative coefficients only. Our reason to mention Conjecture D is its similarity to a former conjecture of Sheil-Small [7] that has been established as a special case of de Branges's proof of the Milin conjecture [1]. That conjecture (now a theorem) reads as follows:

Theorem (de Branges, Sheil-Small). *Let P be a polynomial of degree n. Then for $h \in \mathcal{S}$, we have*

$$\|P * h\| \le n\|P\|. \tag{30}$$

If in (29) we use $P(z) := zF(z)$, for F a polynomial of degree $n-1$, and Bernstein's inequality, then (30) is seen to be a weaker form of (29) (except for the fact that our P has a zero at the origin).

We wish to state Conjecture A in still another equivalent form. To this end we introduce the class $\mathcal{W}$ of functions $w \in \mathcal{A}_0$ satisfying $\|w\| \le 1$ and

$$\frac{|w'(z)|}{1-|w(z)|^2} \le \frac{1}{2}, \qquad z \in \mathbb{D}. \tag{31}$$

Conjecture A″. *Let $w \in \mathcal{W}$, $h \in \mathcal{S}^2$ (or $h \in \mathcal{S}$). Then*

$$\operatorname{Re}\left\{\frac{1}{1-w(z)} * \frac{h(z)}{z}\right\} > \frac{1}{2}, \qquad z \in \mathbb{D}. \tag{32}$$

The relation between $\mathcal{D}'$ and $\mathcal{W}$ is as follows: if

$$G(z) = (1+w(z))/(1-w(z))$$

then $G \in \mathcal{D}'$ iff $w \in \mathcal{W}$. This makes it clear that this latter conjecture is indeed just a reformulation of Conjecture A (and B). However, the connection of $\mathcal{W}$ with the metric in hyperbolic geometry may suggest something.

We close this paper with a general remark. The former conjectures like Bieberbach's, Robertson's, and also Milin's on the class $\mathcal{S}$ were essentially coefficient oriented. Conjecture A is obviously of a different type and is not likely, if it is at all true, to be contained in one of the others, although we are not able to confirm this statement. We admit that this conjecture is a very vulnerable one; however, it covers a vast quantity of known estimates in $\mathcal{S}$ and definitely creates many interesting questions as special cases.

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