EXISTENCE OF SOLUTIONS TO $u'(t) = Au(t)$ FOR A WEAKLY CLOSED

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Abstract. We give an existence theorem of solutions to the evolution equation $(d/dt)u(t) = Au(t)$ for a weakly closed operator $A$ with some conditions on its resolvent in a reflexive Banach space.

Introduction

In this paper we consider the Cauchy problem

\[(CP_T; u_0) \begin{cases} (d/dt)u(t) = Au(t) & \text{for } t \in [0, T), \ 0 < T < \infty, \\ u(0) = u_0 \end{cases} \]

for a given nonlinear operator $A$ in a reflexive Banach space. In [2] we proved the existence of mild solutions to $(CP_T; u_0)$ for an operator $A$ in a Hilbert space under the condition

\[(Ax - A(x - \lambda Ax), \lambda Ax) \leq 0 \]

for $\lambda > 0$, $x \in D(A)$, and $x - \lambda Ax \in D(A)$

and some other conditions. Our purpose is to give an existence theorem of solutions to $(CP_T; u_0)$ for a weakly closed operator $A$ with a weaker condition than (DD). In §1 we state a theorem, and the proofs are given in §2.

1. Notation and theorem

Let $X$ be a reflexive Banach space with norm $\| \|$ and $X^*$ its dual space. We shall denote by $\langle x, x^* \rangle$ the value of $x^* \in X^*$ at $x \in X$. Let $A$ be a single-valued operator in $X$ with domain $D(A)$ and range $R(A)$. We obtain the following

Theorem. Let $1 < p < \infty$ and $0 < T < \infty$. Let $A$ be a single valued operator in $X$ satisfying the following conditions:

(A.1) for every $\lambda > 0$, $(1 - \lambda A)^{-1}$ is a single-valued operator;

(A.2) $R(1 - \lambda A) \supset D(A)$ for all $\lambda > 0$.

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(A.3) for any \( x \in D(A) \), there exists a positive number \( K(x) \) such that
\[
\sum_{k=1}^{n} \left( \frac{n}{T} \right)^{p-1} \left\| \left( 1 - \frac{T}{n} A \right)^{-k} x - \left( 1 - \frac{T}{n} A \right)^{-(k-1)} x \right\|^p \leq K(x)
\]
for \( n = 1, 2, 3, \ldots ; \)

(A.4) \( A \) is a weakly closed operator, i.e., if \( x_n \in D(A) \), \( x_n \to x \) weakly and \( Ax_n \to y \) weakly, then \( x \in D(A) \) and \( Ax = y \).

Under the above conditions, for given \( u_0 \) in \( D(A) \), there exists a Hölder continuous function \( u : [0, T] \to X \) satisfying the equation
\[
u(t) = u_0 + \int_0^t A u(s) \, ds \quad \text{for all } t \in [0, T].
\]

Remark. (DD), (A.1), and (A.2) imply (A.3) by taking \( K(x) = T \|Ax\|^p \). This is easily checked, since (DD) yields the inequality
\[
\|Ax\| \leq \|Ax - \lambda x\|,
\]
and by substituting \( (1 - \lambda A)^{-k} x \) for \( x \) we have
\[
\| (1 - \lambda A)^{-k} x - (1 - \lambda A)^{-(k-1)} x \| \leq \| (1 - \lambda A)^{-(k-1)} x - (1 - \lambda A)^{-(k-2)} x \|.
\]

2. Proof of Theorem

We begin with the following lemma.

Lemma 1. Let \( A \) be a single-valued operator in \( X \) satisfying conditions (A.1)–(A.3) of Theorem. Let \( u_0 \in D(A) \), and put
\[
u^n(t) = \begin{cases} u_k^n & \text{for } t \in \left( \frac{k-1}{n} T, \frac{k}{n} T \right), \\ u_0 & \text{for } t = 0. \end{cases}
\]

Then \( \{u^n\} \) has the following properties:

(i) \( u^n(t) \in D(A) \) for all \( t \in [0, \infty) \), and
\[
u^n(t) = u_0 + \int_0^t A u^n(s) \, ds \quad \text{for } t \in \left( \frac{k-1}{n} T, \frac{k}{n} T \right).
\]

(ii) There exist a subsequence \( \{u^{n_j}\} \) of \( \{u^n\} \) and a function \( u : [0, T] \to X \) such that
\[
\lim_{j \to \infty} u^{n_j}(s) = u(s)
\]
and
\[
\|u(t) - u(s)\| \leq |t - s|^{\frac{1}{2}} \left( K(u_0) \right)^{\frac{1}{2}} \quad \text{for all } t, s \in [0, T],
\]
where \( q = p/(p-1) \).

Proof. It is clear that \( u^n(t) \in D(A) \) for all \( t \in [0, \infty) \). If \( t \in \left( \frac{k-1}{n} T, \frac{k}{n} T \right) \), then
\[
u^n(t) = u_k^n = u_0 + \sum_{j=1}^{k} \left( u_j^n - u_{j-1}^n \right) = u_0 + \sum_{j=1}^{k} \frac{T}{n} A u_j^n
\]
\[
= u_0 + \sum_{j=1}^{k} \int_{j-1}^{j} A u^n(s) \, ds = u_0 + \int_0^t A u^n(s) \, ds.
\]
By virtue of (A.3), we have
\[
\int_0^T \|Au^n(s)\|^p \, ds = \sum_{k=1}^n \int_{k\frac{T}{n}}^{(k+1)\frac{T}{n}} \left\| \frac{n}{T} (u^n_k - u^n_{k-1}) \right\|^p \, ds
\]
(1)
\[
= \sum_{k=1}^n \left( \frac{n}{T} \right)^{p-1} \|u^n_k - u^n_{k-1}\|^p \leq K(u_0).
\]
Since \( \{Au^n\} \) is a bounded sequence in reflexive Banach space \( L^p([0, T]; X) \), there exist a subsequence \( \{Au^{n_j}\} \) of \( \{Au^n\} \) and a function \( v \in L^p([0, T]; X) \) such that
\[
Au^{n_j} \rightharpoonup v \quad \text{weakly in } L^p([0, T]; X).
\]
Therefore for any \( w \in L^q([0, T]; X^*) \), we have
\[
\int_0^T \langle Au^{n_j}(s) - v(s), w(s) \rangle \, ds \to 0 \quad \text{as } j \to \infty,
\]
from which it follows that
\[
\int_0^T \langle Au^{n_j}(s) - v(s), x^* \rangle \, ds \to 0 \quad \text{as } j \to \infty
\]
for all \( t \in [0, T] \) and \( x^* \in X^* \). For \( t \in (\frac{k-1}{n_j} T, \frac{k}{n_j} T) \), we have
\[
u^{n_j}(t) = u_0 + \int_0^t Au^{n_j}(s) \, ds + \int_t^{\frac{k}{n_j} T} Au^{n_j}(s) \, ds.
\]
Using Hölder's inequality and (1), we have
\[
\left\| \int_{\frac{k}{n_j} T}^{\frac{k}{n_j} T} Au^{n_j}(s) \, ds \right\| \leq \int_{\frac{k}{n_j} T}^{\frac{k}{n_j} T} \|Au^{n_j}(s)\| \, ds \leq \left( \frac{k}{n_j} T - t \right)^{\frac{1}{2}} (K(u_0))^{\frac{1}{2}}
\]
(5)
\[
\leq \left( \frac{T}{n_j} \right)^{\frac{1}{2}} (K(u_0))^{\frac{1}{2}} \to 0 \quad \text{as } j \to \infty.
\]
Hence, by (3) and (4),
\[
w- \lim_{j \to \infty} u^{n_j}(t) = u_0 + \int_0^t v(s) \, ds \quad \text{for all } t \in [0, T].
\]
Putting \( u(t) = w- \lim_{j \to \infty} u^{n_j}(t) \), we have, by (2) and (1),
\[
\|v\|_{L^q([0, T]; X)} \leq \lim_{j \to \infty} \|Au^{n_j}\|_{L^p([0, T]; X)} \leq (K(u_0))^\frac{1}{2},
\]
and hence, by (6),
\[
\|u(t) - u(s)\| = \left\| \int_s^t v(s) \, ds \right\| \leq (t-s)^{\frac{1}{2}} (K(u_0))^{\frac{1}{2}}
\]
for \( 0 \leq s < t \leq T \).

**Lemma 2.** Let \( A \) be a single-valued operator satisfying conditions (A.1)-(A.4) of Theorem, and let \( u \) be the weak limit function in (ii). Putting \( S = \{s \in [0, T] ; u(s) \in D(A)\} \), we get the following properties:

(iii) \( S \) is a Lebesgue measurable set and the Lebesgue measure of \( S \), denoted by \( \mu(S) \), is \( \mu(S) = T \).
(iv) \(Au \in L^p(S; X)\), i.e., \(Au(\cdot)\) is strongly measurable in \(S\), and \(\int_S \|Au(s)\|^p \, ds < \infty\).

**Proof.** By Fatou's lemma, we have

\[
\int_0^T \lim_{j \to \infty} \|Au^{n_j}(s)\|^p \, ds \leq \lim_{j \to \infty} \int_0^T \|Au^{n_j}(s)\|^p \, ds \leq K(u_0),
\]

from which it follows that

\[
\alpha(s) \equiv \lim_{j \to \infty} \|Au^{n_j}(s)\| < \infty
\]

for almost all \(s \in [0, T]\). Fix \(s \in [0, T]\) such that \(\alpha(s) < \infty\). Then there exists a subsequence \(\{n_{(j,k)}\}\) of \(\{n_j\}\) such that

\[
\lim_{k \to \infty} \|Au^{n_{(j,k)}}(s)\| = \alpha(s).
\]

For any subsequence \(\{Au^{n_{(j,k,1)}}(s)\}\) of \(\{Au^{n_{(j,k)}}(s)\}\), there exist a subsequence \(\{Au^{n_{(j,k,1,m)}}(s)\}\) of \(\{Au^{n_{(j,k,1)}}(s)\}\) and \(v(s) \in X\) such that

\[
Au^{n_{(j,k,1,m)}}(s) \rightharpoonup v(s) \quad \text{weakly as } m \to \infty.
\]

By virtue of (A.4) and (ii), we have

\[
u(s) \in D(A) \quad \text{and} \quad w- \lim_{k \to \infty} Au^{n_{(j,k,1)}}(s) = Au(s).
\]

Thus \(S\) is a Lebesgue measurable set, and \(\mu(S) = T\). Moreover, we have, by (10), (9), and (8),

\[
\|Au(s)\| \leq \alpha(s) = \lim_{j \to \infty} \|Au^{n_j}(s)\|
\]

for all \(s \in [0, T]\) such that \(\alpha(s) < \infty\). To prove (iv), we set

\[
S_m = \{s \in S; \|Au(s)\| \leq m\}.
\]

First we show that \(S_m\) is a closed set and \(Au(\cdot)\) is weakly continuous on \(S_m\). Assume that \(s_n \in S_m\) and \(s_n \to s\) as \(n \to \infty\). For any subsequence \(\{Au(s_{n_k})\}\) of \(\{Au(s_n)\}\), there exist a subsequence \(\{Au(s_{n_{(k,k)}})\}\) of \(\{Au(s_{n_k})\}\) and \(v(s) \in X\) such that

\[
w- \lim_{k \to \infty} Au(s_{n_{(k,k)}}) = v(s).
\]

By (ii) and (A.4), we have

\[
u(s) \in D(A) \quad \text{and} \quad w- \lim_{n \to \infty} Au(s_n) = Au(s).
\]

Thus

\[
\|Au(s)\| \leq \lim_{n \to \infty} \|Au(s_n)\| \leq m.
\]

Therefore \(S_m\) is a closed set and \(Au(\cdot)\) is weakly continuous on \(S_m\). By standard argument, we can prove that \(Au(\cdot)\) is strongly measurable in \(S_m\) (see [1, p. 73]). Since \(S_m \uparrow S\) as \(m \to \infty\), \(Au(\cdot)\) is strongly measurable in \(S\). Noting (8), we have, by (11) and (7),

\[
\int_S \|Au(s)\|^p \, ds \leq K(u_0). \quad \square
\]
Proof of Theorem. Setting
\[ S_m^n = \{ s \in S ; \| Au^n(s) \| \leq m \} \]
and
\[ v_m^n(s) = \begin{cases} Au^n(s) & \text{if } s \in S_m^n, \\ Au(s) & \text{if } s \in S \setminus S_m^n, \end{cases} \]
we have
\[ \| v_m^n(s) \| \leq \max(m, \| Au(s) \|) \quad \text{for all } s \in S. \]
Moreover, we get, by (ii) and (A.4),
\[ \lim_{j \to \infty} v_m^n(s) = Au(s) \quad \text{for all } s \in S \text{ and } m > 0. \]
By (i) and Lemma 2, for \( t \in (\frac{k-1}{n_j} T, \frac{k}{n_j} T] \),
\[ u^{n_j}(t) = u_0 + \int_0^T Au^{n_j}(s) \, ds \]
\[ = u_0 + \int_0^t v_m^n(s) \, ds + \int_0^t (Au^{n_j}(s) - v_m^n(s)) \, ds \]
\[ + \int_{\frac{k}{n_j} T}^{\frac{k+1}{n_j} T} Au^{n_j}(s) \, ds. \]
By virtue of (13) and (14), using Lebesgue’s dominated convergence theorem, we have
\[ \lim_{j \to \infty} \int_0^t v_m^n(s) \, ds = \int_0^t Au(s) \, ds \quad \text{for } m > 0. \]
Noting that
\[ K(u_0) \geq \int_0^T \| Au^n(s) \|^p \, ds \geq \int_{[0, T] \setminus S_m^n} m^p \, ds = m^p \mu([0, T] \setminus S_m^n) \quad \text{for } m > 0, \]
we have, by (1) and (12),
\[ \left\| \int_0^t (Au^{n_j}(s) - v_m^n(s)) \, ds \right\| \leq \int_{S \setminus S_m^n} \| Au^{n_j}(s) - Au(s) \| \, ds \]
\[ \leq \left( \int_{S \setminus S_m^n} 1 \, ds \right)^{\frac{1}{q}} \left( \int_{S \setminus S_m^n} \| Au^{n_j}(s) - Au(s) \|^p \, ds \right)^{\frac{1}{p}} \]
\[ \leq \left( \mu([0, T] \setminus S_m^n) \right)^{\frac{1}{q}} \left( \int_0^T \| Au^{n_j}(s) \|^p \, ds \right)^{\frac{1}{p}} \]
\[ \leq \left( \frac{K(u_0)}{m^p} \right)^{\frac{1}{q}} \cdot 2(K(u_0))^{\frac{1}{p}}\]
\[ = \frac{2}{m^{p-1}} K(u_0) \to 0 \quad \text{as } m \to \infty. \]
In equation (15), first taking weak limit as \( j \to \infty \) and next letting \( m \to \infty \), by (i), (16), (17), and (5), we have
\[ u(t) = u_0 + \int_0^t Au(s) \, ds \quad \text{for all } t \in [0, T]. \]
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